1. Linear Models
   (a) Growth and Decay
   (b) Half-life of Radioactive
   (c) Carbon Dating
   (d) Newton’s Law of Cooling / Warming
   (e) Chemical Mixtures
   (f) Series Circuit

2. Non-Linear Equations
   (a) Logistics Equation
   (b) Chemical Reactions

3. Systems of Differential Equations
   (a) Radioactive Series
   (b) Mixtures
   (c) Predator-Prey Models
   (d) Competition Models
The initial-value problem

\[
\frac{dx}{dt} = kx, \quad x(t_0) = x_0,
\]

where \( k \) is a constant of proportionality, serves as a model for diverse phenomena involving either growth or decay.
Example - Bacteria Growth

A culture initially has $P_0$ number of bacteria. At $t = 1$ hour, the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time $t$, determine the time necessary for the number of bacteria to triple.

**Answer**

\[
\frac{dP}{dt} = kP, \quad P(0) = P_0, P(1) = \frac{3}{2}P_0.
\]

We can solve this equation using either separation of variables or linear equation. Using separation of variables, we get

\[
P(t) = c e^{kt}
\]

at $t = 0, c = P_0$ so we get

\[
P(t) = P_0 e^{kt}
\]

At $t = 1$, we get

\[
\frac{3}{2}P_0 = P_0 e^k
\]

\[
\frac{3}{2} = e^k
\]

so the general solution is

\[
P(t) = P_0 e^{0.4055 t}
\]

To find the time for bacteria to triple of the initial population, we set

\[
3P_0 = P_0 e^{0.4055 t}
\]

\[
0.4055 t = \ln 3
\]

\[
t = \frac{\ln 3}{0.4055} \approx 2.71 \text{ hours.}
\]
Linear Models - Half-Life

Half-Life is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount of $A_0$ to disintegrate into the atoms of another element.

The longer the half-life of a substance, the more stable it is.
Linear Models - Half-Life

Example - Half-Life of Plutonium

A breeder reactor convert relatively stable uranium 238 into the isotope plutonium 239. After 15 years, it is determined that 0.043% of the initial amount $A_0$ of plutonium has disintegrated.

Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

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**Answer**

The initial-value problem is

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

where $A(t)$ is the amount of plutonium remaining at time $t$. The solution is

$$A(t) = A_0 e^{kt}.$$  

If 0.043% of the atoms of $A_0$ have disintegrated, we still have 99.957% atoms. To find the value for $k$, we use

$$0.99957A_0 = A(15)$$
$$0.99957A_0 = A_0 e^{15k}$$

$$k = \frac{1}{15} \ln 0.99957$$
$$k = -0.00002867$$

Hence

$$A(t) = A_0 e^{-0.00002867t}.$$  

To find the the half-life of plutonium

$$A(t) = \frac{1}{2} A_0$$
$$A_0 e^{-0.00002867t} = \frac{1}{2} A_0$$

$$t = \frac{\ln 2}{0.00002867} \approx 24,180 \text{ years.}$$
A chemist, Willard Libby devised a method of using radioactive carbon as a means of determining the approximate ages of fossils.
Linear Models - Carbon Dating

Example - Age of a Fossil

A fossilized bone is found to contain one-thousandth of the \( C - 14 \) level found in living matter.

Estimate the age of the fossil.

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**Answer**

The initial-value problem is
\[
\frac{dA}{dt} = kA, \quad A(0) = A_0
\]
with the general answer
\[
A(t) = A_0 e^{kt}.
\]

Half-life of \( C - 14 \) is 5600 years, so
\[
\frac{1}{2} A_0 = A(5600) = A_0 e^{5600k} \\
k = \frac{\ln 2}{5600} = -0.00012378 \\
A(t) = A_0 e^{-0.00012378t}.
\]

For this fossil, \( A(t) = \frac{1}{1000} A_0 \), solve for \( t \), we get
\[
\frac{1}{1000} A_0 = A_0 e^{-0.00012378t} \\
t = \frac{\ln 1000}{0.00012378} \\
\approx 55,800 \text{ years.}
\]
Newton’s empirical law of cooling / heating of an object is given by the linear first-order differential equation

$$\frac{dT}{dt} = k(T - T_m)$$

where

- $k$ is constant of proportionality
- $T(t)$ is the temperature of object for $t > 0$
- $T_m$ is the ambient temperature
Linear Models - Newton’s Law of Cooling / Heating

Example - Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F.

How long will it take for the cake to cool off to a room temperature of 70° F?

**Answer**

Our $T_m = 70$, the initial-value problem is

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 300$$

Solving the equation, we get

$$\frac{dT}{T - 70} = k \, dt$$

$$\ln |T - 70| = kt + c_1$$

$$T = 70 + c_2 \, e^{kt}$$

Finding the value for $c_2$ using initial-value $T(0) = 300$, we get

$$300 = 70 + c_2$$

$$c_2 = 230$$

Therefore

$$T(t) = 70 + 230 \, e^{kt}$$

Finding the value for $k$ using $T(3) = 200$, we get

$$T(3) = 200$$

$$e^{3k} = \frac{13}{23}$$

$$k = \frac{1}{3} \ln \frac{13}{23}$$

$$= -0.19018$$

The time for the cake to cool off to room temperature is approximately 30 minutes.
The mixing of two fluids sometimes gives rise to a linear first-order differential equation.

If the amount of salt in the mixing tanks net rate is $A'(t)$, then

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt}) = R_{in} - R_{out}$$
Linear Models - Mixtures

Example - Mixing of Two Salt Solutions

When we do the mixing, the information that we have are

\[ R_{in} = (2 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = 6 \text{ lb/min} \]

\[ R_{out} = \left( \frac{A}{300} \text{ lb/gal} \right) \cdot (3 \text{ gal/min}) = \frac{A}{100} \text{ lb/min} \]

If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time?

**Answer**

The initial-value problem is

\[ \frac{dA}{dt} = 6 - \frac{A}{100}, \quad A(0) = 50. \]

Treating this equation as linear first-order DE, the general solution is

\[ A(t) = 600 - c e^{-t/100} \]

Using initial-value, \( A(0) = 50 \), to find \( c \), we get \( c = -550 \), so the particular solution is

\[ A(t) = 600 - 550 e^{-t/100} \]

When \( t \to \infty \), we get

\[ A(t) = 600 \text{ pounds.} \]
For series circuit containing only a resistor and an inductor, Kirchhoff’s second law states that the sum of the voltage drop across the inductor and the voltage drop across the resistor is the same as the impressed voltage on the circuit.

\[ L \frac{di}{dt} + Ri = E(t). \]

And for series circuit containing only a resistor and a capacitor

\[ R \frac{dq}{dt} + \frac{1}{C} q = E(t). \]
A 12-volt battery is connected to a series circuit in which the inductance is $1/2$ Henry and the resistance is 10 Ohms. Determine the current, $i$, if the initial current is zero.

The equation to be solve is

$$\frac{d}{dt} \left( \frac{1}{2} i \right) + 10i = 12, \quad i(0) = 0.$$

Using linear first-order method, the general solution is

$$i(t) = \frac{6}{5} + ce^{-20t}.$$

Using initial condition, $i(0) = 0$, we find the value for $c$

$$0 = \frac{6}{5} + c \quad \Rightarrow \quad c = -\frac{6}{5}.$$

So the particular solution is

$$i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}.$$
1. The population of a community is known to increase at a rate proportional to the number of people present at time $t$.
   If an initial population $P_0$ has doubled in 5 years, how long will it take to triple? To quadruple?

2. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time $t$ and has a half-life of 3.3 hours.
   If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

3. Archaeologists used pieces of burned wood, or charcoal, found at the site to date prehistoric paintings and drawings on walls and ceilings of a cave in Lascaux, France.
   Determine the approximate age of a piece of burned wood, if it was found that 85.5% of the C-14 found in living trees of the same type had decayed.
4. A thermometer is removed from a room where the temperature is 70° F and is taken outside, where the air temperature is 10° F. After one-half minute the thermometer reads 50° F.

What is the reading of the thermometer at $t = 1$ min?

How long will it take for the thermometer to reach 15° F?

5. A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of grams of salt in the tank at time $t$.

6. A 30-volt electromotive force is applied to an $LR$-series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms.

Find the current $i(t)$ if $i(0) = 0$.

Determine the current as $t \to \infty$. 
Answers for Exercises

1. 7.9 Years; 10 years.
2. 11 hours.
3. 15,600 years.
4. $T(1) = 36.67^\circ F$; approximately 3.06 min.
5. $A(t) = 200 - 170 e^{-t/50}$
6. $i(t) = \frac{3}{5} - \frac{3}{5} e^{-500t}; i \rightarrow \frac{3}{5}$ as $t \rightarrow \infty$