INSTRUCTIONS:

- Answer the following questions.
- Check your answer for odd number questions at the back of the textbook.

1.1 Exercises
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1.2 Exercises
1, 3, 5, 7, 9, 11, 13

2.2 Exercises
1, 3, 5, 7, 13, 15, 17, 19, 21, 25, 34, 35

2.3 Exercises
7, 9, 13, 15, 17, 19, 25(a), 37
Although the majority of equations one is likely to encounter in practice fall into the non-linear category, knowing how to deal with the simpler linear equations is an important first step (just as tangent lines help our understanding of complicated curves by providing local approximations).

### 1.1 Exercises

In Problems 1–12, a differential equation is given along with the field or problem area in which it arises. Classify each as an ordinary differential equation (ODE) or a partial differential equation (PDE), give the order, and indicate the independent and dependent variables. If the equation is an ordinary differential equation (ODE), indicate whether the equation is linear or nonlinear.

1. \( \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0 \)  
   (Hermite’s equation, quantum-mechanical harmonic oscillator)

2. \( 5 \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 9x = 2 \cos 3t \)  
   (mechanical vibrations, electrical circuits, seismology)

3. \( \frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = 0 \)  
   (Laplace’s equation, potential theory, electricity, heat, aerodynamics)

4. \( \frac{dy}{dx} = \frac{y(2 - 3x)}{x(1 - 3y)} \)  
   (competition between two species, ecology)

5. \( \frac{dx}{dt} = k(4 - x)(1 - x) \), where \( k \) is a constant  
   (chemical reaction rates)

6. \( y \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] = C \), where \( C \) is a constant  
   (brachistochrone problem,\(^1\) calculus of variations)

7. \( \sqrt{1 - y} \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0 \)  
   (Kidder’s equation, flow of gases through a porous medium)

8. \( \frac{dp}{dt} = kp(P - p) \), where \( k \) and \( P \) are constants  
   (logistic curve, epidemiology, economics)

9. \( 8 \frac{d^4y}{dx^4} = x(1 - x) \)  
   (deflection of beams)

10. \( \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \)  
    (aerodynamics, stress analysis)

11. \( \frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r} \frac{\partial N}{\partial r} + kN \), where \( k \) is a constant  
    (nuclear fission)

12. \( \frac{d^2y}{dx^2} - 0.1(1 - y^2) \frac{dy}{dx} + 9y = 0 \)  
    (van der Pol’s equation, triode vacuum tube)

In Problems 13–16, write a differential equation that fits the physical description.

13. The rate of change of the population \( P \) of bacteria at time \( t \) is proportional to the population at time \( t \).

14. The velocity at time \( t \) of a particle moving along a straight line is proportional to the fourth power of its position \( x \).

15. The rate of change in the temperature \( T \) of coffee at time \( t \) is proportional to the difference between the temperature \( M \) of the air at time \( t \) and the temperature of the coffee at time \( t \).

16. The rate of change of the mass \( A \) of salt at time \( t \) is proportional to the square of the mass of salt present at time \( t \).

17. **Drag Race.** Two drivers, Alison and Kevin, are participating in a drag race. Beginning from a standing start, they each proceed with a constant acceleration. Alison covers the last 1/4 of the distance in 3 seconds, whereas Kevin covers the last 1/3 of the distance in 4 seconds. Who wins and by how much time?

---

\(^1\)**Historical Footnote:** In 1630 Galileo formulated the brachistochrone problem (\( \beta \rho \alpha \chi \iota \sigma \rho \alpha \sigma \varsigma = \) shortest, \( \chi \rho \omicron \nu \omicron \sigma \varsigma = \) time), that is, to determine a path down which a particle will fall from one given point to another in the shortest time. It was reproduced by John Bernoulli in 1696 and solved by him the following year.
Example 8  For the initial value problem

\[(11) \quad \frac{dy}{dx} = x^2 - xy^3, \quad y(1) = 6 \]

does Theorem 1 imply the existence of a unique solution?

**Solution**  Dividing by 3 to conform to the statement of the theorem, we identify \(f(x, y) = (x^2 - xy^3)/3\) and \(\partial f/\partial y = -xy^2\). Both of these functions are continuous in any rectangle containing the point \((1, 6)\), so the hypotheses of Theorem 1 are satisfied. It then follows from the theorem that the initial value problem \((11)\) has a unique solution in an interval about \(x = 1\) of the form \((1 - \delta, 1 + \delta)\), where \(\delta\) is some positive number. 

Example 9  For the initial value problem

\[(12) \quad \frac{dy}{dx} = 3y^{2/3}, \quad y(2) = 0 \]

does Theorem 1 imply the existence of a unique solution?

**Solution**  Here \(f(x, y) = 3y^{2/3}\) and \(\partial f/\partial y = 2y^{-1/3}\). Unfortunately \(\partial f/\partial y\) is not continuous or even defined when \(y = 0\). Consequently, there is no rectangle containing \((2, 0)\) in which both \(f\) and \(\partial f/\partial y\) are continuous. Because the hypotheses of Theorem 1 do not hold, we cannot use Theorem 1 to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution. We refer you to Problem 29 and Group Project G of Chapter 2 for the details.

In Example 9 suppose the initial condition is changed to \(y(2) = 1\). Then, since \(f\) and \(\partial f/\partial y\) are continuous in any rectangle that contains the point \((2, 1)\) but does not intersect the \(x\)-axis—say, \(R = \{(x, y): 0 < x < 10, 0 < y < 5\}\)—it follows from Theorem 1 that this new initial value problem has a unique solution in some interval about \(x = 2\).

### 1.2 Exercises

1. (a) Show that \(y^2 + x - 3 = 0\) is an implicit solution to \(dy/dx = -1/(2y)\) on the interval \((-\infty, 3)\).
   
   (b) Show that \(xy^3 - xy^3\sin x = 1\) is an implicit solution to
   
   \[
   \frac{dy}{dx} = \frac{(x \cos x + \sin x - 1)y}{3(x - x \sin x)}
   \]
   
   on the interval \((0, \pi/2)\).

2. (a) Show that \(\phi(x) = x^2\) is an explicit solution to
   
   \[
   \frac{dy}{dx} = 2y
   \]
   
   on the interval \((-\infty, \infty)\).
   
   (b) Show that \(\phi(x) = e^x - x\) is an explicit solution to
   
   \[
   \frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1
   \]
   
   on the interval \((-\infty, \infty)\).

3. (c) Show that \(\phi(x) = x^2 - x^{-1}\) is an explicit solution to \(x^2dy/dx^2 = 2y\) on the interval \((0, \infty)\).

   In Problems 3–8, determine whether the given function is a solution to the given differential equation.

3. \(x = 2 \cos t - 3 \sin t, \quad x'' + x = 0\)

4. \(y = \sin x + x^2, \quad \frac{d^2y}{dx^2} + y = x^2 + 2\)

5. \(x = \cos 2t, \quad \frac{dx}{dt} + tx = \sin 2t\)

6. \(\theta = 2e^{3t} - e^{2t}, \quad \frac{d^2\theta}{dt^2} - \theta \frac{d\theta}{dt} + 3\theta = -2e^{2t}\)

7. \(y = 3 \sin 2x + e^{-x}, \quad y'' + 4y = 5e^{-x}\)

8. \(y = e^{2x} - 3e^{-x}, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y = 0\)
In Problems 9–13, determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define \( y \) implicitly as a function of \( x \) and use implicit differentiation.

9. \( y - \ln y = x^2 + 1 \), \( \frac{dy}{dx} = \frac{2xy}{y-1} \)

10. \( x^2 + y^2 = 4 \), \( \frac{dy}{dx} = \frac{x}{y} \)

11. \( e^{xy} + y = x - 1 \), \( \frac{dy}{dx} = \frac{e^{-xy} - y}{e^{-xy} + x} \)

12. \( x^2 - \sin(x + y) = 1 \), \( \frac{dy}{dx} = 2x \sec(x + y) - 1 \)

13. \( \sin y + xy - x^3 = 2 \), \( y' = \frac{6xy' + (y')^3 \sin y - 2(y')^2}{3x^2 - y} \)

14. Show that \( \phi(x) = c_1 \sin x + c_2 \cos x \) is a solution to \( d^2y/dx^2 + y = 0 \) for any choice of the constants \( c_1 \) and \( c_2 \). Thus, \( c_1 \sin x + c_2 \cos x \) is a two-parameter family of solutions to the differential equation.

15. Verify that \( \phi(x) = \sqrt{1 - c^2} \), where \( c \) is an arbitrary constant, is a one-parameter family of solutions to \( \frac{dy}{dx} = \frac{y(y-2)}{2} \).

Graph the solution curves corresponding to \( c = 0, \pm 1, \pm 2 \) using the same coordinate axes.

16. Verify that \( x^2 + cy^2 = 1 \), where \( c \) is an arbitrary nonzero constant, is a one-parameter family of implicit solutions to \( \frac{dy}{dx} = \frac{xy}{x^2 - 1} \) and graph several of the solution curves using the same coordinate axes.

17. Show that \( \phi(x) = Ce^{3x} + 1 \) is a solution to \( dy/dx - 3y = -3 \) for any choice of the constant \( C \). Thus, \( Ce^{3x} + 1 \) is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

18. Let \( c > 0 \). Show that the function \( \phi(x) = (c^2 - x^2)^{-1} \) is a solution to the initial value problem \( dy/dx = 2xy^2 \), \( y(0) = 1/c^2 \), on the interval \(-c < x < c\). Note that this solution becomes unbounded as \( x \) approaches \( \pm c \). Thus, the solution exists on the interval \((-\delta, \delta)\) with \( \delta = c \), but not for larger \( \delta \). This illustrates that in Theorem 1 the existence interval can be quite small (if \( c \) is small) or quite large (if \( c \) is large). Notice also that there is no clue from the equation \( dy/dx = 2xy^2 \) itself, or from the initial value, that the solution will "blow up" at \( x = \pm c \).

19. Show that the equation \((dy/dx)^2 + y^2 + 4 = 0\) has no (real-valued) solution.

20. Determine for which values of \( m \) the function \( \phi(x) = e^{mx} \) is a solution to the given equation.

(a) \( \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 5y = 0 \)

(b) \( \frac{d^3y}{dx^3} + 3 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0 \)

21. Determine for which values of \( m \) the function \( \phi(x) = x^m \) is a solution to the given equation.

(a) \( 3x^2 \frac{d^2y}{dx^2} + 11x \frac{dy}{dx} - 3y = 0 \)

(b) \( x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 5y = 0 \)

22. Verify that the function \( \phi(x) = c_1 e^x + c_2 e^{-2x} \) is a solution to the linear equation \( \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \) for any choice of the constants \( c_1 \) and \( c_2 \). Determine \( c_1 \) and \( c_2 \) so that each of the following initial conditions is satisfied.

(a) \( y(0) = 2 \), \( y'(0) = 1 \)

(b) \( y(1) = 1 \), \( y'(1) = 0 \)

In Problems 23–28, determine whether Theorem 1 implies that the given initial value problem has a unique solution.

23. \( \frac{dy}{dx} = y^4 - x^4 \), \( y(0) = 7 \)

24. \( \frac{dy}{dt} - ty = \sin^2 t \), \( y(\pi) = 5 \)

25. \( 3x \frac{dx}{dt} + 4t = 0 \), \( x(2) = -\pi \)

26. \( \frac{dx}{dt} + \cos x = \sin t \), \( x(\pi) = 0 \)

27. \( \frac{dy}{dx} = x \), \( y(1) = 0 \)

28. \( \frac{dy}{dx} = 3x - \sqrt{y - 1} \), \( y(2) = 1 \)
2.2 EXERCISES

In Problems 1–6, determine whether the given differential equation is separable.

1. \( \frac{dy}{dx} - \sin(x + y) = 0 \)  
2. \( \frac{dy}{dx} = 4y^2 - 3y + 1 \)

3. \( \frac{ds}{dt} = t \ln(s^2) + 8t^2 \)  
4. \( \frac{dy}{dx} = \frac{ye^{x+y}}{x^2 + 2} \)

5. \( (xy^2 + 3y^3)dy - 2x \, dx = 0 \)  
6. \( s^2 + \frac{ds}{dt} = \frac{s + 1}{st} \)

In Problems 7–16, solve the equation.

7. \( \frac{dx}{dt} = 3xt^2 \)  
8. \( \frac{dy}{dx} = \frac{1}{y^3} \)

9. \( \frac{dy}{dx} = \frac{x}{y^2\sqrt{1 + x}} \)  
10. \( \frac{dx}{dt} = \frac{t}{xe^{2x}} \)

11. \( \frac{dy}{dx} = \frac{\sec^2y}{1 + x^2} \)  
12. \( \frac{dx}{dt} = \frac{1 - 4u^2}{3u} \)

13. \( \frac{dx}{dt} = x^3 = x \)  
14. \( \frac{dy}{dx} = 3x^2(1 + y^2)^{3/2} \)

15. \( y^{-1} \, dy + ye^{\cos x} \sin x \, dx = 0 \)  
16. \( (x + xy^2)dx + e^xy \, dy = 0 \)

In Problems 17–26, solve the initial value problem.

17. \( y' = x^3(1 - y) \), \( y(0) = 3 \)  
18. \( \frac{dy}{dx} = (1 + y^2)\tan x \), \( y(0) = \sqrt{3} \)

19. \( \frac{1}{2} \, \frac{dy}{dx} = \sqrt{y + 1} \cos x \), \( y(\pi) = 0 \)  
20. \( \frac{dx}{dy} = 4x^2 - x - 2 \) \( (x + 1)(y + 1) \), \( y(1) = 1 \)

21. \( \frac{1}{\theta} \, \frac{dy}{d\theta} = \frac{y \sin \theta}{y^2 + 1} \), \( y(\pi) = 1 \)  
22. \( x^2 \, dx + 2y \, dy = 0 \), \( y(0) = 2 \)

23. \( \frac{t^{-1} \, dy}{dt} = 2 \cos^2 y \), \( y(0) = \pi/4 \)  
24. \( \frac{dy}{dx} = 8x^3e^{-2y} \), \( y(1) = 0 \)

25. \( \frac{dy}{dx} = x^2(1 + y) \), \( y(0) = 3 \)  
26. \( \sqrt{y} \, dx + (1 + x) \, dy = 0 \), \( y(0) = 1 \)

27. **Solutions Not Expressible in Terms of Elementary Functions.** As discussed in calculus, certain indefinite integrals (antiderivatives) such as \( \int e^x \, dx \) cannot be expressed in finite terms using elementary functions. When such an integral is encountered while solving a differential equation, it is often helpful to use definite integration (integrals with variable upper limit). For example, consider the initial value problem

\[ \frac{dy}{dx} = e^{x^2}, \quad y(2) = 1. \]

The differential equation separates if we divide by \( y^2 \) and multiply by \( dx \). We integrate the separated equation from \( x = 2 \) to \( x = x_1 \) and find

\[ \int_{x=2}^{x=x_1} e^{x^2} \, dx = \int_{x=2}^{x=x_1} \frac{dy}{y^2}, \]

\[ = -\frac{1}{y} \bigg|_{x=2}^{x=x_1}, \]

\[ = -\frac{1}{y(x_1) + \frac{1}{y(2)}}. \]

If we let \( t \) be the variable of integration and replace \( x_1 \) by \( x \) and \( y(2) \) by 1, then we can express the solution to the initial value problem by

\[ y(x) = \left(1 - \int_2^x e^{t^2} \, dt\right)^{-1}. \]

Use definite integration to find an explicit solution to the initial value problems in parts (a)–(c).

(a) \( \frac{dy}{dx} = e^{x^2}, \quad y(0) = 0 \)

(b) \( \frac{dy}{dx} = e^{x^2}y^{-2}, \quad y(0) = 1 \)

(c) \( \frac{dy}{dx} = \sqrt{1 + \sin x} (1 + y^2), \quad y(0) = 1 \)

(d) Use a numerical integration algorithm (such as Simpson’s rule, described in Appendix C) to approximate the solution to part (b) at \( x = 0.5 \) to three decimal places.

28. Sketch the solution to the initial value problem

\[ \frac{dy}{dt} = 2y - 2yt, \quad y(0) = 3 \]

and determine its maximum value.

29. **Uniqueness Questions.** In Chapter 1 we indicated that in applications most initial value problems will have a unique solution. In fact, the existence of unique
solutions was so important that we stated an existence and uniqueness theorem, Theorem 1, page 11. The method for separable equations can give us a solution, but it may not give us all the solutions (also see Problem 30). To illustrate this, consider the equation \( dy/dx = y^{1/3} \).

(a) Use the method of separation of variables to show that
\[
y = \left( \frac{2x}{3} + C \right)^{3/2}
\]
is a solution.

(b) Show that the initial value problem \( dy/dx = y^{1/3} \) with \( y(0) = 0 \) is satisfied for \( C = 0 \) by \( y = (2x/3)^{3/2} \) for \( x \geq 0 \).

(c) Now show that the constant function \( y = 0 \) also satisfies the initial value problem given in part (b). Hence, this initial value problem does not have a unique solution.

(d) Finally, show that the conditions of Theorem 1 on page 11 are not satisfied.

(The solution \( y = 0 \) was lost because of the division by zero in the separation process.)

30. As stated in this section, the separation of equation (2) on page 39 requires division by \( p(y) \), and this may disguise the fact that the roots of the equation \( p(y) = 0 \) are actually constant solutions to the differential equation.

(a) To explore this further, separate the equation
\[
\frac{dy}{dx} = (x - 3)(y + 1)^{2/3}
\]
to derive the solution,
\[
y = -1 + \left( \frac{x^2}{6} - x + C \right)^3.
\]

(b) Show that \( y = -1 \) satisfies the original equation \( dy/dx = (x - 3)(y + 1)^{2/3} \).

(c) Show that there is no choice of the constant \( C \) that will make the solution in part (a) yield the solution \( y = -1 \). Thus, we lost the solution \( y = -1 \) when we divided by \((y + 1)^{2/3}\).

31. Interval of Definition. By looking at an initial value problem \( dy/dx = f(x,y) \) with \( y(x_0) = y_0 \), it is not always possible to determine the domain of the solution \( y(x) \) or the interval over which the function \( y(x) \) satisfies the differential equation.

(a) Solve the equation \( dy/dx = xy^3 \).

(b) Give explicitly the solutions to the initial value problem with \( y(0) = 1 \); \( y(0) = 1/2 \); \( y(0) = 2 \).

(c) Determine the domains of the solutions in part (b).

(d) As found in part (c), the domains of the solutions depend on the initial conditions. For the initial value problem \( dy/dx = xy^3 \) with \( y(0) = a \), \( a > 0 \), show that as \( a \) approaches zero from the right the domain approaches the whole real line \((-\infty, \infty)\) and as \( a \) approaches \(+\infty\) the domain shrinks to a single point.

(e) Sketch the solutions to the initial value problem \( dy/dx = xy^3 \) with \( y(0) = a \) for \( a = \pm 1/2, \pm 1, \) and \( \pm 2 \).

32. Analyze the solution \( y = \phi(x) \) to the initial value problem
\[
\frac{dy}{dx} = y^2 - 3y + 2, \quad y(0) = 1.5
\]
using approximation methods and then compare with its exact form as follows.

(a) Sketch the direction field of the differential equation and use it to guess the value of \( \lim_{x \to \infty} \phi(x) \).

(b) Use Euler’s method with a step size of 0.1 to find an approximation of \( \phi(1) \).

(c) Find a formula for \( \phi(x) \) and graph \( \phi(x) \) on the direction field from part (a).

(d) What is the exact value of \( \phi(1) \)? Compare with your approximation in part (b).

(e) Using the exact solution obtained in part (c), determine \( \lim_{x \to \infty} \phi(x) \) and compare with your guess in part (a).

33. Mixing. Suppose a brine containing 0.3 kilogram (kg) of salt per liter (L) runs into a tank initially filled with 400 L of water containing 2 kg of salt. If the brine enters at 10 L/min, the mixture is kept uniform by stirring, and the mixture flows out at the same rate. Find the mass of salt in the tank after 10 min (see Figure 2.4). [Hint: Let \( A \) denote the number of kilograms of salt in the tank at \( t \) min after the process begins and use the fact that rate of increase in \( A = \) rate of input \(-\) rate of exit.]

A further discussion of mixing problems is given in Section 3.2.]

Figure 2.4 Schematic representation of a mixing problem
34. Newton’s Law of Cooling. According to Newton’s law of cooling, if an object at temperature \( T \) is immersed in a medium having the constant temperature \( M \), then the rate of change of \( T \) is proportional to the difference of temperature \( M - T \). This gives the differential equation

\[
\frac{dT}{dt} = k(M - T)
\]

(a) Solve the differential equation for \( T \).
(b) A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 min, the thermometer reads 80°F. What is the reading after 20 min?

(Further applications of Newton’s law of cooling appear in Section 3.3.)

35. Blood plasma is stored at 40°F. Before the plasma can be used, it must be at 90°F. When the plasma is placed in an oven at 120°F, it takes 45 min for the plasma to warm to 90°F. Assume Newton’s law of cooling (Problem 34) applies. How long will it take for the plasma to warm to 90°F if the oven temperature is set at (a) 100°F, (b) 140°F, and (c) 80°F?

36. A pot of boiling water at 100°C is removed from a stove at time \( t = 0 \) and left to cool in the kitchen. After 5 min, the water temperature has decreased to 80°C, and another 5 min later it has dropped to 65°C. Assuming Newton’s law of cooling (Problem 34) applies, determine the (constant) temperature of the kitchen.

37. Compound Interest. If \( P(t) \) is the amount of dollars in a savings bank account that pays a yearly interest rate of \( r\% \) compounded continuously, then

\[
\frac{dP}{dt} = \frac{r}{100} P, \quad t \text{ in years.}
\]

Assume the interest is 5% annually, \( P(0) = 1000 \), and no monies are withdrawn.

(a) How much will be in the account after 2 yr?
(b) When will the account reach $4000?
(c) If $1000 is added to the account every 12 months, how much will be in the account after 3\( \frac{1}{2} \) yr?

38. Free Fall. In Section 2.1, we discussed a model for an object falling toward Earth. Assuming that only air resistance and gravity are acting on the object, we found that the velocity \( v \) must satisfy the equation

\[
m \frac{dv}{dt} = mg - bv,
\]

where \( m \) is the mass, \( g \) is the acceleration due to gravity, and \( b > 0 \) is a constant (see Figure 2.1).

If \( m = 100 \text{ kg}, g = 9.8 \text{ m/sec}^2, b = 5 \text{ kg/sec}, \) and \( v(0) = 10 \text{ m/sec} \), solve for \( v(t) \). What is the limiting (i.e., terminal) velocity of the object?

39. Grand Prix Race. Driver A had been leading archrival B for a while by a steady 3 miles. Only 2 miles from the finish, driver A ran out of gas and decelerated thereafter at a rate proportional to the square of his remaining speed. One mile later, driver A’s speed was exactly halved. If driver B’s speed remained constant, who won the race?

40. The atmospheric pressure (force per unit area) on a surface at an altitude \( z \) is due to the weight of the column of air situated above the surface. Therefore, the difference in air pressure \( p \) between the top and bottom of a cylindrical volume element of height \( \Delta z \) and cross-section area \( A \) equals the weight of the air enclosed (density \( \rho \), times volume \( V = A \Delta z \), times gravity \( g \)), per unit area:

\[
p(z + \Delta z) - p(z) = -\frac{\rho(z)(A \Delta z)g}{A} = -\rho(z) g \Delta z.
\]

Let \( \Delta z \to 0 \) to derive the differential equation \( dp/dz = -pg \). To analyze this further we must postulate a formula that relates pressure and density. The perfect gas law relates pressure, volume, mass \( m \), and absolute temperature \( T \) according to \( pV = mRT/M \), where \( R \) is the universal gas constant and \( M \) is the molar mass of the air. Therefore, density and pressure are related by \( \rho := m/V = Mfp/RT \).

(a) Derive the equation \( \frac{dp}{dz} = -\frac{Mg}{RT} p \) and solve it for the “isothermal” case where \( T \) is constant to obtain the barometric pressure equation \( p(z) = p(z_0) \exp[-Mg(z-z_0)/RT] \).
(b) If the temperature also varies with altitude \( T = T(z) \), derive the solution

\[
p(z) = p(z_0) \exp\left\{ -\frac{Mg}{R} \int_{z_0}^{z} \frac{dz}{T(z)} \right\}.
\]

(c) Suppose an engineer measures the barometric pressure at the top of a building to be 99,000 Pa (pascals), and 101,000 Pa at the base \( (z = z_0) \). If the absolute temperature varies as \( T(z) = 288 - 0.0065(z - z_0) \), determine the height of the building. Take \( R \) as 8.31 N-m/mol-K, \( M \) as 0.029 kg/mol, and \( g \) as 9.8 m/sec\(^2\). (An amusing story concerning this problem can be found at http://www.snopes.com/college/exam/barometer.asp)
In Example 3 we had no difficulty expressing the integral for the integrating factor $\mu(x) = e^{\int p(x)\,dx} = e^t$. Clearly, situations will arise where this integral, too, cannot be expressed with elementary functions. In such cases we must again resort to a numerical procedure such as Euler’s method (Section 1.4) or to a “nested loop” implementation of Simpson’s rule. You are invited to explore such a possibility in Problem 27.

Because we have established explicit formulas for the solutions to linear first-order differential equations, we get as a dividend a direct proof of the following theorem.

The essentials of the proof of Theorem 1 are contained in the deliberations leading to equation (8); Problem 34 provides the details. This theorem differs from Theorem 1 on page 11 in that for the linear initial value problem (15), we have the existence and uniqueness of the solution on the whole interval $(a, b)$, rather than on some smaller unspecified interval about $x_0$.

The theory of linear differential equations is an important branch of mathematics not only because these equations occur in applications but also because of the elegant structure associated with them. For example, first-order linear equations always have a general solution given by equation (8). Some further properties of first-order linear equations are described in Problems 28 and 36. Higher-order linear equations are treated in Chapters 4, 6, and 8.

2.3 Exercises

In Problems 1–6, determine whether the given equation is separable, linear, neither, or both.

1. $\frac{dx}{dt} + xt = e^x$
2. $x^2 \frac{dy}{dx} + \sin x - y = 0$
3. $3t = e^{\frac{dy}{dt}} + y \ln t$
4. $(t^2 + 1) \frac{dy}{dt} = yt - y$
5. $3r = \frac{dr}{d\theta} - \theta^3$
6. $x \frac{dy}{dx} + t^2 = \sin t$

In Problems 7–16, obtain the general solution to the equation.

7. $\frac{dy}{dx} = \frac{y}{x} + 2x + 1$
8. $\frac{dy}{dx} - y - e^{3x} = 0$
9. $\frac{dy}{dx} + 2y = x^{-3}$
10. $\frac{dr}{d\theta} + r \tan \theta = \sec \theta$
11. $(t + y + 1) \frac{dy}{dt} - dy = 0$
12. $\frac{dy}{dx} = x^2e^{-4x} - 4y$
13. $\frac{dx}{dy} + 2x = 5y^3$
14. $\frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x}$
15. $(x^2 + 1) \frac{dy}{dx} + xy - x = 0$
16. $(1 - x^2) \frac{dy}{dx} - x^2y = (1 + x) \sqrt{1 - x^2}$
In Problems 17–22, solve the initial value problem.

17. \( \frac{dy}{dx} - \frac{y}{x} = xe^x \), \( y(1) = e - 1 \)

18. \( \frac{dy}{dx} + 4y - e^{-x} = 0 \), \( y(0) = \frac{4}{3} \)

19. \( t \frac{dx}{dt} + 3tx = t^4 \ln t + 1 \), \( x(1) = 0 \)

20. \( \frac{dy}{dx} + \frac{3y}{x} + 2 = 3x \), \( y(1) = 1 \)

21. \( \cos x \frac{dy}{dx} + y \sin x = 2x \cos^2 x \),
   \( y\left(\frac{\pi}{4}\right) = -15 \sqrt{2} \pi^2 / 32 \)

22. \( \sin x \frac{dy}{dx} + y \cos x = x \sin x \), \( y\left(\frac{\pi}{2}\right) = 2 \)

23. Radioactive Decay. In Example 2 assume that the rate at which \( RA_1 \) decays into \( RA_2 \) is \( 40e^{-20t} \) kg/sec and the decay constant for \( RA_2 \) is \( k = 5 \) sec. Find the mass \( y(t) \) of \( RA_2 \) for \( t \geq 0 \) if initially \( y(0) = 10 \) kg.

24. In Example 2 the decay constant for isotope \( RA_1 \) was \( 10 \) sec, which expresses itself in the exponent of the rate term \( 50e^{-10t} \) kg/sec. When the decay constant for \( RA_2 \) is \( k = 2 \) sec, we see that in formula (14) for \( y \) the term \( (185/4)e^{-2t} \) eventually dominates (has greater magnitude for \( t \) large).

(a) Redo Example 2 taking \( k = 20 \) sec. Now which term in the solution eventually dominates?

(b) Redo Example 2 taking \( k = 10 \) sec.

25. (a) Using definite integration, show that the solution to the initial value problem
   \( \frac{dy}{dx} + 2xy = 1 \), \( y(2) = 1 \),
   can be expressed as
   \( y(x) = e^{-x^2} \left( e^x + \int_{2}^{x} e^{t^2} \, dt \right) \).
   (b) Use numerical integration (such as Simpson’s rule, Appendix C) to approximate the solution at \( x = 3 \).

26. Use numerical integration (such as Simpson’s rule, Appendix C) to approximate the solution, at \( x = 1 \), to the initial value problem
   \( \frac{dy}{dx} + \frac{\sin 2x}{2(1 + \sin^2 x)} y = 1 \), \( y(0) = 0 \).
   Ensure your approximation is accurate to three decimal places.

27. Consider the initial value problem
   \( \frac{dy}{dx} + \sqrt{1 + \sin^2 x} y = x \), \( y(0) = 2 \).

(a) Using definite integration, show that the integrating factor for the differential equation can be written as
   \( \mu(x) = \exp \left( \int_{0}^{x} \sqrt{1 + \sin^2 t} \, dt \right) \)
   and that the solution to the initial value problem is
   \( y(x) = \frac{1}{\mu(x)} \int_{0}^{x} \mu(s) \, s \, ds + \frac{2}{\mu(x)} \).
   (b) Obtain an approximation to the solution at \( x = 1 \) by using numerical integration (such as Simpson’s rule, Appendix C) in a nested loop to estimate values of \( \mu(x) \) and, thereby, the value of
   \( \int_{0}^{1} \mu(s) \, ds \).

   [Hint: First, use Simpson’s rule to approximate \( \mu(x) \) at \( x = 0.1, 0.2, \ldots, 1 \). Then use these values and apply Simpson’s rule again to approximate \( \int_{0}^{1} \mu(s) \, ds \).]

(c) Use Euler’s method (Section 1.4) to approximate the solution at \( x = 1 \), with step sizes \( h = 0.1 \) and \( 0.05 \).

[A direct comparison of the merits of the two numerical schemes in parts (b) and (c) is very complicated, since it should take into account the number of functional evaluations in each algorithm as well as the inherent accuracies.]

28. Constant Multiples of Solutions.

(a) Show that \( y = e^{-x} \) is a solution of the linear equation
   \( \frac{dy}{dx} + y = 0 \),
   and \( y = x^{-1} \) is a solution of the nonlinear equation
   \( \frac{dy}{dx} + y^2 = 0 \).

(b) Show that for any constant \( C \), the function \( Ce^{-x} \)
   is a solution of equation (16), while \( Cx^{-1} \) is a solution of equation (17) only when \( C = 0 \) or 1.

(c) Show that for any linear equation of the form
   \( \frac{dy}{dx} + p(x)y = 0 \),
   if \( \tilde{y}(x) \) is a solution, then for any constant \( C \) the function \( C\tilde{y}(x) \) is also a solution.
29. Use your ingenuity to solve the equation
\[
\frac{dy}{dx} = \frac{1}{e^{2y} + 2x}.
\]
[Hint: The roles of the independent and dependent variables may be reversed.]

30. Bernoulli Equations. The equation
\[
(18) \quad \frac{dy}{dx} + 2y = xy^{-2}
\]
is an example of a Bernoulli equation. (Further discussion of Bernoulli equations is in Section 2.6.)

(a) Show that the substitution \( v = y^3 \) reduces equation (18) to the equation
\[
(19) \quad \frac{dv}{dx} + 6v = 3x.
\]

(b) Solve equation (19) for \( v \). Then make the substitution \( v = y^3 \) to obtain the solution to equation (18).

31. Discontinuous Coefficients. As we will see in Chapter 3, occasions arise when the coefficient \( P(x) \) in a linear equation fails to be continuous because of jump discontinuities. Fortunately, we may still obtain a “reasonable” solution. For example, consider the initial value problem
\[
\frac{dy}{dx} + P(x)y = x, \quad y(0) = 1,
\]
where
\[
P(x) := \begin{cases} 
1, & 0 \leq x \leq 2, \\
3, & x > 2.
\end{cases}
\]

(a) Find the general solution for \( 0 \leq x \leq 2 \).
(b) Choose the constant in the solution of part (a) so that the initial condition is satisfied.
(c) Find the general solution for \( x > 2 \).
(d) Now choose the constant in the general solution from part (c) so that the solution from part (b) and the solution from part (c) agree at \( x = 2 \). By patching the two solutions together, we can obtain a continuous function that satisfies the differential equation except at \( x = 2 \), where its derivative is undefined.
(e) Sketch the graph of the solution from \( x = 0 \) to \( x = 5 \).

32. Discontinuous Forcing Terms. There are occasions when the forcing term \( Q(x) \) in a linear equation fails to be continuous because of jump discontinuities. Fortunately, we may still obtain a reasonable solution imitating the procedure discussed in Problem 31. Use this procedure to find the continuous solution to the initial value problem.
\[
\frac{dy}{dx} + 2y = Q(x), \quad y(0) = 0,
\]
where
\[
Q(x) := \begin{cases} 
2, & 0 \leq x \leq 3, \\
-2, & x > 3.
\end{cases}
\]
Sketch the graph of the solution from \( x = 0 \) to \( x = 7 \).

33. Singular Points. Those values of \( x \) for which \( P(x) \) in equation (4) is not defined are called singular points of the equation. For example, \( x = 0 \) is a singular point of the equation \( xy' + 2y = 3x \), since when the equation is written in the standard form, \( y' + (2/x)y = 3 \), we see that \( P(x) = 2/x \) is not defined at \( x = 0 \). On an interval containing a singular point, the questions of the existence and uniqueness of a solution are left unanswered, since Theorem 1 does not apply. To show the possible behavior of solutions near a singular point, consider the following equations.

(a) Show that \( xy' + 2y = 3x \) has only one solution defined at \( x = 0 \). Then show that the initial value problem for this equation with initial condition \( y(0) = y_0 \) has a unique solution when \( y_0 = 0 \) and no solution when \( y_0 \neq 0 \).
(b) Show that \( xy' - 2y = 3x \) has an infinite number of solutions defined at \( x = 0 \). Then show that the initial value problem for this equation with initial condition \( y(0) = 0 \) has an infinite number of solutions.

34. Existence and Uniqueness. Under the assumptions of Theorem 1, we will prove that equation (8) gives a solution to equation (4) on \( [a, b] \). We can then choose the constant \( C \) in equation (8) so that the initial value problem (15) is solved.

(a) Show that since \( P(x) \) is continuous on \( [a, b] \), then \( \mu(x) \) defined in (7) is a positive, continuous function satisfying \( d\mu/dx = P(x)\mu(x) \) on \( [a, b] \).
(b) Since
\[
\frac{d}{dx} \int_a^x \mu(x)Q(x) \, dx = \mu(x)Q(x),
\]
verify that \( y \) given in equation (8) satisfies equation (4) by differentiating both sides of equation (8).
(c) Show that when we let \( \int_a^x \mu(x)Q(x) \, dx \) be the antiderivative whose value at \( x_0 \) is 0 (i.e., \( \int_a^t \mu(t)Q(t) \, dt \) and choose \( C \) to be \( y_0 \mu(x_0) \), the initial condition \( y(x_0) = y_0 \) is satisfied.
(d) Start with the assumption that $y(x)$ is a solution to the initial value problem (15) and argue that the discussion leading to equation (8) implies that $y(x)$ must obey equation (8). Then argue that the initial condition in (15) determines the constant $C$ uniquely.

35. Mixing. Suppose a brine containing 0.2 kg of salt per liter runs into a tank initially filled with 500 L of water containing 5 kg of salt. The brine enters the tank at a rate of 5 L/min. The mixture, kept uniform by stirring, is flowing out at the rate of 5 L/min (see Figure 2.6).

![Figure 2.6 Mixing problem with equal flow rates](image)

(a) Find the concentration, in kilograms per liter, of salt in the tank after 10 min. [Hint: Let $A$ denote the number of kilograms of salt in the tank at $t$ minutes after the process begins and use the fact that rate of increase in $A = \text{rate of input} - \text{rate of exit}$.

A further discussion of mixing problems is given in Section 3.2.]

(b) After 10 min, a leak develops in the tank and an additional liter per minute of mixture flows out of the tank (see Figure 2.7). What will be the concentration, in kilograms per liter, of salt in the tank 20 min after the leak develops? [Hint: Use the method discussed in Problems 31 and 32.]

36. Variation of Parameters. Here is another procedure for solving linear equations that is particularly useful for higher-order linear equations. This method is called variation of parameters. It is based on the idea that just by knowing the form of the solution, we can substitute into the given equation and solve for any unknowns. Here we illustrate the method for first-order equations (see Sections 4.6 and 6.4 for the generalization to higher-order equations).

(a) Show that the general solution to

\[
(20) \quad \frac{dy}{dx} + P(x)y = Q(x)
\]

has the form

\[
y(x) = Cy_h(x) + y_p(x),
\]

where $y_h (\neq 0)$ is a solution to equation (20) when $Q(x) = 0$, $C$ is a constant, and $y_p(x) = v(x)y_h(x)$ for a suitable function $v(x)$. [Hint: Show that we can take $y_h = \mu^{-1}(x)$ and then use equation (8).]

We can in fact determine the unknown function $y_h$ by solving a separable equation. Then direct substitution of $vC$ in the original equation will give a simple equation that can be solved for $v$.

Use this procedure to find the general solution to

\[
(21) \quad \frac{dy}{dx} + \frac{3}{x}y = x^2, \quad x > 0
\]

by completing the following steps:

(b) Find a nontrivial solution $y_h$ to the separable equation

\[
(22) \quad \frac{dy}{dx} + \frac{3}{x}y = 0, \quad x > 0
\]

(c) Assuming (21) has a solution of the form $y_h(x) = v(x)y_0(x)$, substitute this into equation (21), and simplify to obtain $\nu'(x) = x^2/y_0(x)$.

(d) Now integrate to get $\nu(x)$.

(e) Verify that $y(x) = Cy_h(x) + v(x)y_0(x)$ is a general solution to (21).

37. Secretion of Hormones. The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-h cycle, then the rate of change of the level of the hormone in the blood may be represented by the initial value problem

\[
\frac{dx}{dt} = \alpha - \beta \cos \frac{\pi t}{12} - kx, \quad x(0) = x_0
\]

where $x(t)$ is the amount of the hormone in the blood at time $t$, $\alpha$ is the average secretion rate, $\beta$ is the amount of daily variation in the secretion, and $k$ is a positive constant reflecting the rate at which the body removes the hormone from the blood. If $\alpha = \beta = 1$, $k = 2$, and $x_0 = 10$, solve for $x(t)$.