Linear Congruences

**Question 1.** \(\diamondsuit\) \(a \equiv 2 \pmod{5}\) is the class of \(\{\ldots, -3, 2, 7, 12, \ldots\}\). Based on the same class, express this class using modulo 15.

**Question 2.** \(\diamondsuit\) Write the following linear congruences as linear diophantine equation.

(a) \(4x \equiv 7 \pmod{9}\)  
(b) \(3x \equiv 49 \pmod{10}\)  
(c) \(21x \equiv 6 \pmod{45}\)  
(d) \(14x \equiv 7 \pmod{6}\)

**Question 3.** \(\diamondsuit\) If \(x \equiv 2 \pmod{5}\), then \(x \equiv \) (mod 15).

**Question 4.** For each of the following linear congruences, is there any solution? One class? Multiple classes or no solution?

(a) \(2x \equiv 5 \pmod{7}\)  
(b) \(9x \equiv 5 \pmod{25}\)  
(c) \(3x \equiv 6 \pmod{9}\)  
(d) \(5x \equiv 1 \pmod{10}\)

**Question 5.** Find all solutions of each of the following linear congruences.

(a) \(3x \equiv 2 \pmod{7}\)  
(b) \(6x \equiv 3 \pmod{9}\)  
(c) \(17x \equiv 14 \pmod{21}\)  
(d) \(42x \equiv 90 \pmod{156}\)  
(e) \(19x \equiv 30 \pmod{40}\)  
(f) \(128x \equiv 833 \pmod{1001}\)

**Question 6.** \(\diamondsuit\) Find an inverse modulo 13 of each of the following integers.

(a) 2  
(b) 4  
(c) 5  
(d) 3  
(e) 11

**Question 7.** Find the inverse of 19 (mod 43) using Euclidean algorithm. (To find the inverse of \(x \pmod{m}\), solve \(yx + km = 1\). Thus \(y\) is the inverse of \(x\).)

**Question 8.** For each of the following linear Diophantine equations determine if it has solutions, and if so determine all the solutions:

(a) \(4x + 51y = 9\)  
(b) \(84x + 436y = 156\)  
(c) \(172x + 20y = 1000\)

\([4x \equiv 9 \pmod{51}\] gives \(x = 15 + 51t\), whereas \(51y \equiv 9 \pmod{4}\) gives \(y = 3 + 4s\). Then find relationship between \(s\) and \(t\)].

**Question 9.** An astronomer knows that a satellite orbits the Earth in a period that is an exact multiple of 1 hour that is less than 1 day. If the astronomer notes that the satellite completes 11 orbits in an interval that starts when a 24-hour clock reads 0 hours and ends when the clock reads 17 hours, how long is the orbital period of the satellite?

**Question 10.** You started a long mathematics exam at 2:00 p.m.. You were told that you could work as long as you liked. You worked 487 hours straight. When did you finish? You must indicate morning or afternoon.

**Question 11.** Show that if \(a\) is an inverse of \(a \pmod{m}\) and \(b\) is an inverse of \(b \pmod{m}\), then \(ab\) is an inverse of \(ab \pmod{m}\).

**Chinese Remainder Theorem**

**Question 12.** \(\diamondsuit\) Solve the linear congruences below.

(a) \(3x + 4 \equiv 7 \pmod{7}\)  
(b) \(7 + 6x \equiv 9 \pmod{5}\)  
(c) \(6 \equiv 3x - 5 \pmod{7}\)
Question 13. [☐] Find the inverse of 2 for following modulo

(a) \(2x \equiv 1 \pmod{7}\)  \hspace{1cm} (b) \(2x \equiv 1 \pmod{3}\)  \hspace{1cm} (c) \(2x \equiv 1 \pmod{11}\)

Question 14. Find an integer that leaves a remainder of 1 when divided by either 2 or 5, but that is divisible by 3.

Question 15. Find all the solutions of each of the following system of linear congruences.

\[
\begin{align*}
(a) \quad x &\equiv 4 \pmod{11} \\
&\equiv 3 \pmod{17}
\end{align*}
\quad
\begin{align*}
(b) \quad x &\equiv 1 \pmod{2} \\
&\equiv 2 \pmod{3} \\
&\equiv 3 \pmod{5}
\end{align*}
\quad
\begin{align*}
(c) \quad x &\equiv 0 \pmod{2} \\
&\equiv 0 \pmod{3} \\
&\equiv 1 \pmod{5} \\
&\equiv 6 \pmod{7}
\end{align*}
\]

Question 16. A troop of 17 monkeys store their bananas in 11 piles of equal size with a twelfth pile of 6 left over. When they divide the bananas into 17 equals group, none remain. What is the smallest number of bananas they can have?

Question 17. Find all the solutions to the system of linear congruences \(x \equiv 2 \pmod{11}\), \(7x \equiv 4 \pmod{12}\) and \(x \equiv 4 \pmod{13}\).

Question 18. A general initially has 500 troops. After the battle, if there were 2 left over when they lined up 5 at a time, 3 left over when they lined up 6 at a time, and none left over when they lined up 11 at a time, how many troops remained after the battle? (Answer: 297 troops)

Question 19. Al Haytam considered a problem in “Opuscula”: To find a number such that if we divide by two, one remains; if we divide by three, one remains; if we divide by four, one remains; if we divide by five, one remains; if we divide by six, one remains; if we divide by seven, there is no remainder. Solve this problem.

Question 20. [*] Find all the solutions of each of the following system of linear congruences.

\[
\begin{align*}
(a) \quad x &\equiv 4 \pmod{6} \\
&\equiv 13 \pmod{15}
\end{align*}
\quad
\begin{align*}
(b) \quad x &\equiv 7 \pmod{10} \\
&\equiv 4 \pmod{15}
\end{align*}
\]

[Hint: Write the first congruence as \(x = a_1 + km_1\), and then insert this expression for \(x\) into second congruences]

Question 21. Find an integer that leaves a remainder of 9 when it is divided by either 10 or 11, but that is divisible by 13.

Question 22. [Brahmagupta, 7th century A.D.] When eggs in a basket are removed 2, 3, 4, 5, 6 at a time there remain, respectively, 1,2,3,4,5 eggs. When there are taken out 7 at a time, none are left over. Find the smallest number of eggs that could have been contained in the basket.

Question 23. Find the smallest integer \(a > 2\) such that \(2|a\), \(3|a + 1\), \(4|a + 2\), \(5|a + 3\), \(6|a + 4\).

Question 24. Given \(47 \equiv 1 \pmod{2}\) and \(47 \equiv 2 \pmod{3}\), what is the remainder of when 47 divide by 6?

Question 25. What is the remainder of 4589 when divide by 99 by considering 4589 \(\pmod{11}\) and 4589 \(\pmod{9}\)?

Question 26. Solve this linear congruence \(42x \equiv 90 \pmod{156}\) by CRT where 156 = \(2^2 \cdot 3 \cdot 13\).