Linear Congruences

Question 1. If \( x \equiv 2 \pmod{5} \), then \( x \equiv ? \pmod{15} \).

Question 2. For each of the following linear congruences, is there any solution? One class? Multiple classes or no solution?

(a) \( 2x \equiv 5 \pmod{7} \)  
(b) \( 9x \equiv 5 \pmod{25} \)  
(c) \( 3x \equiv 6 \pmod{9} \)

Question 3. Find all solutions of each of the following linear congruences.

(a) \( 3x \equiv 2 \pmod{7} \)  
(b) \( 6x \equiv 3 \pmod{9} \)  
(c) \( 17x \equiv 14 \pmod{21} \)  
(d) \( 19x \equiv 30 \pmod{40} \)

Question 4. Find an inverse modulo 13 of each of the following integers.

(a) 2  
(b) 4  
(c) 5  
(d) 3  
(e) 11

Question 5. Find the inverse of 19 (mod 43) using Euclidean algorithm. (To find the inverse of \( x \pmod{m} \), solve \( yx + km = 1 \). Thus \( y \) is the inverse of \( x \).)

Question 6. For each of the following linear Diophantine equations determine if it has solutions, and if so determine all the solutions:

(a) \( 4x + 51y = 9 \)  
(b) \( 84x + 436y = 156 \)  
(c) \( 172x + 20y = 1000 \)

\[ 4x \equiv 9 \pmod{51} \] gives \( x = 15 + 51t \), whereas \( 51y \equiv 9 \pmod{4} \) gives \( y = 3 + 4s \). Then find relationship between \( s \) and \( t \).

Question 7. An astronomer knows that a satellite orbits the Earth in a period that is an exact multiple of 1 hour that is less than 1 day. If the astronomer notes that the satellite completes 11 orbits in an interval that starts when a 24-hour clock reads 0 hours and ends when the clock reads 17 hours, how long is the orbital period of the satellite?

Question 8. Show that if \( \bar{a} \) is an inverse of \( a \) modulo \( m \) and \( \bar{b} \) is an inverse of \( b \) modulo \( m \), then \( \bar{a} \bar{b} \) is an inverse of \( ab \) modulo \( m \).

Question 9. A customer bought some apples and some oranges, 12 pieces of fruit in total, and they cost him RM1.32. If an apple costs 3 cents more than an orange, and if more apples than oranges were purchased, how many pieces of each fruit were bought?

Chinese Remainder Theorem

Question 10. Solve the linear congruences below.

(a) \( 3x + 4 \equiv 7 \pmod{7} \)  
(b) \( 7 + 6x \equiv 9 \pmod{5} \)  
(c) \( 6 \equiv 3x - 5 \pmod{7} \)

Question 11. Find the inverse of 2 for following modulo

(a) \( 2x \equiv 1 \pmod{7} \)  
(b) \( 2x \equiv 1 \pmod{3} \)  
(c) \( 2x \equiv 1 \pmod{11} \)

Question 12. Find an integer that leaves a remainder of 1 when divided by either 2 or 5, but that is divisible by 3.
Question 13. Find all the solutions of each of the following system of linear congruences.

(a) \( x \equiv 4 \pmod{11} \)
\( x \equiv 3 \pmod{17} \)

(b) \( x \equiv 1 \pmod{2} \)
\( x \equiv 2 \pmod{3} \)
\( x \equiv 3 \pmod{5} \)

(c) \( x \equiv 0 \pmod{2} \)
\( x \equiv 0 \pmod{3} \)
\( x \equiv 1 \pmod{5} \)
\( x \equiv 6 \pmod{7} \)

Question 14. A troop of 17 monkeys store their bananas in 11 piles of equal size with a twelfth pile of 6 left over. When they divide the bananas into 17 equals group, none remain. What is the smallest number of bananas they can have?

Question 15. Find all the solutions to the system of linear congruences \( x \equiv 2 \pmod{11} \), \( 7x \equiv 4 \pmod{12} \) and \( x \equiv 4 \pmod{13} \).

Question 16. Al Haytam considered a problem in “Opuscula”: To find a number such that if we divide by two, one remains; if we divide by three, one remains; if we divide by four, one remains; if we divide by five, one remains; if we divide by six, one remains; if we divide by seven, there is no remainder. Solve this problem.

Question 17. Find an integer that leaves a remainder of 9 when it is divided by either 10 or 11, but that is divisible by 13.

Question 18. [Brahmagupta, 7th century A.D.] When eggs in a basket are removed 2, 3, 4, 5, 6 at a time there remain, respectively, 1, 2, 3, 4, 5 eggs. When there are taken out 7 at a time, none are left over. Find the smallest number of eggs that could have been contained in the basket.

Question 19. Find the smallest integer \( a > 2 \) such that \( 2|a \), \( 3|a + 1 \), \( 4|a + 2 \), \( 5|a + 3 \), \( 6|a + 4 \).

Question 20. Given \( 47 \equiv 1 \pmod{2} \) and \( 47 \equiv 2 \pmod{3} \), what is the remainder of when 47 divide by 6?

Question 21. What is the remainder of 4589 when divide by 99 by considering 4589 (mod 11) and 4589 (mod 9)?