1. Find the derivative of the function \( f(x) = \begin{cases} 
2x & \text{if } x < 0 \\
2x - 1 & \text{if } x \geq 0 
\end{cases} \)

Solution:
The function is given by two linear polynomials but on different intervals. It is clear that the function is not continuous at \( x=0 \), because \( \lim_{x \to 0^-} f(x) = 0 \neq \lim_{x \to 0^+} f(x) = -1 \). Thus, it is not differentiable at \( x = 0 \). The derivative is then given by

\[
f'(x) = \begin{cases} 
2 & \text{if } x \neq 0 \\
doesn't exit & \text{if } x = 0 
\end{cases}
\]

2. Evaluate \( \int_{-1}^{1} f(x) \, dx \) where \( f(x) = \begin{cases} 
x^2 & \text{if } x < 0 \\
x^3 + 2 & \text{if } x \geq 0 
\end{cases} \)

Solution: To evaluate the definite integral of the piecewise-defined function we split the definite integral into:

\[
\int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} \frac{x^2}{x^3 + 2} \, dx + \int_{0}^{1} x^3 \, dx
\]

\[
= \left[ \frac{1}{3} \ln(x^3 + 2) \right]_{-1}^{0} + \left[ \frac{1}{3} x^3 \right]_{0}^{1}
\]

\[
= \frac{1}{3} \ln(2) + \frac{1}{3}
\]

3. Sketch the graph of the function \( f(x) = \ln(x^2 + 1) \).

You have to get: domain, limits, asymptotes, c.n., increasing and decreasing intervals, local extrema, concavity.

Use \( \ln(2) = 0.7, \ln(3) = 1.1, \ln(5) = 1.6 \).
Solution:
Domain: \(( -\infty, \infty )\)
Limits: \( \lim_{t \to \pm\infty} f(x) = \infty \)
There is no HA, VA, or SA.
Derivative: \( f'(x) = \frac{2x}{x^2 + 1} \)
CN: \( f'(x) = 0 \) leads to \( x = 0 \), the only critical number.
The function \( f \) decreases on \( x < 0 \) and increases on \( x > 0 \).
Using first derivative test we conclude that \( f(0)=0 \) is a local minimum.
\[
f''(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2}
\]
\( f''(x) = 0 \) when \( x = \pm 1 \)
\( f \) is concave up on \(( -1, 1)\) and concave down on \(( -\infty, -1) \cup (1, \infty)\)
\( f \) has two inflection points, \(( \pm 1, \ln(2))\).
4. Compute the following integrals:

(i) \[ \int_1^2 \left( \sqrt{x} + \frac{1}{x^2} \right) \, dx \]

(ii) \[ \int \cot x \, dx \]

Solution:

\[
\int_1^2 \left( \sqrt{x} + \frac{1}{x^2} \right) \, dx = \frac{2}{3} x^{3/2} - \frac{1}{x} \Bigr|_1^2 \\
= \frac{1}{6} + \frac{4}{3} \sqrt{2}
\]

\[
\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx \\
= \ln |\sin x| + c
\]

5. Consider the parabola \( y = x^2 \). Show that \( B = 2A \), where:

- **A** is the area between the parabola, the \( y \)-axis and the tangent line at \( x_0 \).
- **B** is the area between the parabola, the \( y \)-axis and the horizontal line \( y = x_0^2 \).
Solution:
Without loss of generality let assume that $x_0$ is positive. The equation of the tangent line at $(x_0, x_0^2)$ is given by

$$y = (2x_0)(x - x_0) + x_0^2$$

The area between the parabola, the $y$-axis and the tangent line at $x_0$ is given by

$$A = \int_0^{x_0} x^2 - [(2x_0)(x - x_0) + x_0^2] \, dx$$

$$= \int_0^{x_0} (x^2 - 2x_0x + x_0^2) \, dx$$

$$= \frac{1}{3}x_0^3$$

The area between the parabola, the $y$-axis and the horizontal line $y = x_0^2$ is given by

$$B = \int_0^{x_0} x_0^2 - x^2 \, dx$$

$$= \frac{2}{3}x_0^3$$

Therefore, $B = 2A$. 