Solutions of Final Exam

Semester 1 2011/2012 Calculus I

CSC 1705

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Consider the function

\[ f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \]

(i) Is \( f \) continuous at \( x = 0 \)? (3 marks)

(ii) Is \( f \) differentiable at \( x = 0 \)? (5 marks)

Solution:

(i) We know that

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x}, \]

\[ = 1, \]

\[ = f(0). \]

Therefore, the function is continuous at \( x = 0 \).

(ii) Now we compute the derivative at \( x = 0 \)

\[ f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}, \]

\[ = \lim_{h \to 0} \frac{\sin h - 1}{h}, \]

\[ = \lim_{h \to 0} \frac{\sin(h) - h}{h^2}. \]

Using the l’Hopital’s rule we get

\[ f'(0) = \lim_{h \to 0} \frac{\cos h - 1}{2h}, \]

\[ = \lim_{h \to 0} -\frac{\sin h}{2}, \]

\[ = 0, \text{ the limit exists.} \]

Thus, the function is differentiable at \( x = 0 \).

Q.E.D.
For \( f(x) = \sin x \), find \( f^{(150)}(x) \). (5 marks)

**Solution:** Let find few derivatives of our function \( f \).

\[
\begin{align*}
f^{(0)}(x) &= \sin x \\
f^{(1)}(x) &= \cos x \\
f^{(2)}(x) &= -\sin x \\
f^{(3)}(x) &= -\cos x
\end{align*}
\]

Computing the fourth derivative we get

\( f^{(4)}(x) = \sin x = f^{(0)}(x) \).

Thus, the derivative operation is now periodic of period four. That is

\[
f^{(4n+r)}(x) = f^{(r)}(x),
\]

where \( n \) is a nonnegative integer and \( 0 \leq r \leq 3 \). So, to find the 150th derivative we simply use

\[
150 = 4 \times 37 + 2 \quad \text{or} \quad 150 \equiv 2[4].
\]

which leads to

\[
f^{(150)}(x) = f^{(2)}(x) = -\sin x.
\]

Q.E.D.
Consider the following function

\[ f(x) = \frac{1}{x - 1}. \]

(i) Sketch the graph of the function \( f \).

You have to get: domain, limits, asymptotes, c.n., increasing and decreasing, local extrema, concavity. (5 marks)

(ii) The area bounded by the tangent line to the curve, the \(-x\)-axis and the vertical asymptote is constant. Find this constant. (5 marks)

(iii) Find the closest point on the graph to the point \((1, 0)\). (5 marks)

Solution:

(i)

Domain : \((-\infty, 1) \cup (1, \infty)\)

Limits :
\[
\lim_{t \to \pm \infty} f(x) = 0, \quad \text{H. A.} \quad y = 0
\]
\[
\lim_{t \to 1^\pm} f(x) = \pm \infty, \quad \text{V. A.} \quad x = 1
\]

\[
f'(x) = -\frac{1}{(x - 1)^2} < 0, \quad \text{decreasing function}
\]

: no critical number, no local extrema

\[
f''(x) = \frac{2}{(x - 1)^3}
\]

concave up \( x \in (1, \infty) \)

concave down \( x \in (-\infty, 1) \)

The graph of the function is shown in Fig. 3.1.

(ii) The slope at any point \( x_0 \neq 1 \) is given by

\[
m = f'(x_0) = -\frac{1}{(x_0 - 1)^2}
\]
Figure 3.1: The graph of the function $f(x) = 1/(x - 1)$. The dashed area is the area calculated in (ii), and the circle is to show the closest point on the graph to the point (1,0).

The equation of the tangent line is

$$y = m(x - x_0) + \frac{1}{x_0 - 1}$$

$$= -\frac{x - x_0}{(x_0 - 1)^2} + \frac{1}{x_0 - 1}$$

$$= -\frac{x + x_0 + x_0 - 1}{(x_0 - 1)^2}$$

$$= -\frac{x + 2x_0 - 1}{(x_0 - 1)^2}$$

The $x$-intercept for this tangent line can be found at

$$y = 0 \Rightarrow x_a = 2x_0 - 1.$$  

To find the area bounded by the tangent line, the $x$-axis and the vertical asymptote $y = 1$, we evaluate the following integral\(^1\)

\(^1\)In the figure 3.1 we are using $x_0 > 1$. Similarly, for $x_0 < 1$, the area is given by $A = \int_{x_0}^{1} -y \, dx$ which equals to $\int_{1}^{x_0} y \, dx$. 
\[ A = \int_1^{2x_0-1} y \, dx \]
\[ = \int_1^{2x_0-1} \frac{-x + 2x_0 - 1}{(x_0 - 1)^2} \, dx \]
\[ = -\frac{x^2}{2} + \frac{2x_0 - x}{(x_0 - 1)^2} \bigg|_1^{2x_0-1} \]
\[ = \left[ \frac{-2x_0 + 2(x_0 - 1)^2}{2(x_0 - 1)^2} \right] - \left[ \frac{-1/2 + (2x_0 - 1)}{(x_0 - 1)^2} \right] \]
\[ = \frac{2x_0 - 1)^2 + 3 - 4x_0}{2(x_0 - 1)^2} \]
\[ = \frac{4x_0^2 - 4x_0 + 1}{2(x_0 - 1)^2} \]
\[ = \frac{4(x_0 - 1)^2}{2(x_0 - 1)^2} \]
\[ = 2 \]

**Alternative solution:**

Altemately, we can use the area of the triangle shown in Fig. 3.1. We substituting \( x = 1 \) in \( y \).

\[
x = 1 \Rightarrow y_a = \frac{-2 + 2x_0}{(x_0 - 1)^2} \cdot \frac{2}{x_0 - 1}
\]

Therefore the area is given by the area of the triangle

\[
A = \frac{1}{2} (x_a - 1) \times y_a
\]
\[
= \frac{1}{2} (2x_0 - 1 - 1) \times \frac{2}{x_0 - 1}
\]
\[
= \frac{1}{2} 2(x_0 - 1) \times \frac{2}{x_0 - 1}
\]
\[
= 2.
\]

\[ A = 2 \]

(iii) We can get the distance as follows

\[
d^2 = (x - 1)^2 + y^2
\]
\[
= (x - 1)^2 + \frac{1}{(x - 1)^2}
\]

\[ ^2 \text{We are using } x_0 > 1. \text{ In similar way we obtain the same result for } x_0 < 1. \]
We get the minimum of the function $d^2$ as follows

\[
(d^2)' = 2(x - 1) - \frac{2}{(x - 1)^3} = 2(x - 1)^4 - 1 \quad \frac{(x - 1)^2 + 1}{(x - 1)^3} = 2 \left( \frac{(x - 1)^2 + 1}{(x - 1)^3} \right)^2
\]

Now, we look for the zeros of the previous equation. we get

\[(x - 1)^2 - 1 = 0 \Rightarrow x = 0 \quad \text{or} \quad x = 2.
\]

We substitute the values obtained into the function $f(x)$

\[x = 0, f(0) = 1 \quad \text{and} \quad x = 2, f(2) = 1.
\]

We have two points $(0,1)$ and $(2,1)$ (see Fig. 3.1).

Q.E.D.
Compute the following integrals

(i) \( \int_1^2 \left( \sqrt{x} - \frac{1}{x} \right) \, dx \) (2 marks)

(ii) \( \int \cos \sqrt{x} \, dx \) (4 marks)

(iii) \( \int \tan x \, dx \) (2 marks)

Solution

(i) 
\[
\int_1^2 \left( \sqrt{x} - \frac{1}{x} \right) \, dx = \left. \frac{2}{3} x^{3/2} - \ln |x| \right|_1^2
= \left[ \frac{4}{3} \sqrt{2} - \ln 2 \right] - \left[ \frac{2}{3} - \ln 1 \right]
= \frac{4}{3} \sqrt{2} - \frac{2}{3} - \ln 2.
\]

(ii) We use the new variable \( \mu = \sqrt{x} \) then 
\[
d\mu = \frac{1}{2\sqrt{x}} \, dx.
\]
Thus,
\[
\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx = 2 \int \cos \mu \, d\mu
= 2 \sin \mu + c = 2 \sin \sqrt{x} + c
\]

(iii) 
\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln |\cos x| + c.
\]
Q.E.D.
5 Question 5

Find the area between the curves on the given interval

\[ y = \cos x, \quad y = x^2 + 2, \quad 0 \leq x \leq 2. \]

(5 marks)

Solution \hspace{1em} \text{It is clear that the cosine function is always below } y = x^2 + 2, \text{ because } \cos x \leq 1 \text{ and } x^2 + 2 > 1. \text{ So, there is no intersection between the two graphs. Thus, the area between the two graphs is simply}

\[
\int_{0}^{2} x^2 + 2 - \cos x \, dx = \left. \frac{x^3}{3} + 2x - \sin x \right|_{0}^{2}
\]

\[
= \left[ \frac{8}{3} + 4 - \sin 2 \right]
\]

\[
= \frac{20}{3} - \sin 2.
\]

Q.E.D.