6.1 ROTATIONAL MOTION AND ANGULAR DISPLACEMENT.

Angular displacement (analogous to linear displacement)

\[ \Delta \theta = \theta - \theta_0 \]

In radian measure, the angle, \( \theta \) is defined to be:

\[ \theta = \frac{s}{r} \]

where:

- \( s \) = arc length, the distance traveled along the circular path.
- \( r \) = radius of the circle.

One revolution, the arc length,

\[ s = 2\pi r \text{ (circumference of the circle)} \]

From \( \theta = \frac{s}{r} \)

\[ 2\pi r / r = 360^\circ = 2\pi \text{ rad.} \]

1 rad = \[ \frac{360}{2\pi} = 57.3^\circ. \]

\[ \theta \text{(rad)} = \frac{\pi}{180} \theta \text{(deg)} \]

**E.g.**

\[ 60^\circ = \pi/3 \text{ rad} \]
\[ 45^\circ = \pi/4 \text{ rad} \]
EXAMPLE 6.1:
The tires on a new compact car have a diameter of 2.0 ft and are warranted for 60 000 miles.
a) Determine the angle (in radians) through which one of these tires will rotate during the warranty period?
b) How many revolutions of the tire are equivalent to your answer in (a)?

Solution:
a) \[ \theta = \frac{s}{r} = \frac{60000\text{mi}}{1.0\text{ft}} = \left( \frac{60000\text{mi}}{1.0\text{ft}} \right) \left( \frac{5280\text{ft}}{1.0\text{mi}} \right) = 3.2 \times 10^8 \text{ rad} \]
b) \[ \theta = 3.2 \times 10^8 \text{ rad} = 3.2 \times 10^8 \text{ rad} \left( \frac{1\text{rev}}{2\pi\text{rad}} \right) = 5.0 \times 10^7 \text{ rev} \]

6.2 ANGULAR VELOCITY AND ANGULAR ACCELERATION.

The **average angular velocity**, \( \omega \), of a rotating rigid object is the ratio of the angular displacement to the time interval

\[ \omega = \frac{\Delta \theta}{\Delta t} = \frac{\theta - \theta_0}{t - t_0} \]

**Instantaneous angular velocity** (linear speed)
-defined as the limit of the average speed as the time interval approaches zero.

\[ \omega = \lim \frac{\Delta \theta}{\Delta t} \]

Unit : rad/s or rpm (revolution per minute)

\( \omega \) +ve \( \rightarrow \) \( \theta \) is increasing (CCW motion)

\( \omega \) -ve \( \rightarrow \) \( \theta \) is decreasing (CW motion)

EXAMPLE 6.2:
The rotor on a helicopter turns at an angular speed of 320 rev/min. Express this in rad/second.

Solution:
1 rev=2 π rad and 60s= 1 min
320rev/min = 320 (rev/m) (2π rad/1rev)(1min/60s)
= 10.7 π rad/s

**Average angular acceleration**, \( \bar{\alpha} \)

\( \bar{\alpha} = \) the ratio of the change in the angular speed to the time it takes the object to undergo the change.
\[-\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta \omega}{\Delta t}\]

**EXAMPLE 6.4:**
A bicycle tire turning at 0.21 rad/s is brought to rest by the brakes in exactly two revolutions. What is the angular acceleration of the wheel?

Solution:
Given: \(\omega_o = 0.21 \text{ rad/s}\), \(\omega = 0 \text{ rad/s}\), \(\theta_i = 0 \text{ rad}\), \(\theta_f = 4\pi \text{ rad}\),

By using \(\omega^2 = \omega_o^2 + 2 \alpha \theta\),

the angular acceleration of the wheel is given by

\[\alpha = -1.8 \times 10^{-3} \text{ rad/s}^2.\]

**EXAMPLE 6.5:**
The wheel on a moving car slows uniformly from 70 rad/s to 42 rad/s in 4.2 s.

a) What is the angular acceleration of the wheel?

b) What angle does the wheel turn through in the 4.2 s?

c) How far does the car go if the radius of the wheel is 0.32 m?

Answer:

a) -6.7 rad/s²

b) 240 rad

c) 75 m

### 6.3 THE EQUATIONS OF ROTATIONAL KINEMATICS.

<table>
<thead>
<tr>
<th>Linear motion</th>
<th>Rotational motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-v = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t})</td>
<td>(\omega = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t})</td>
</tr>
<tr>
<td>(-a = \frac{v_f - v_i}{t_f - t_i} = \frac{\Delta v}{\Delta t})</td>
<td>(-\alpha = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta \omega}{\Delta t})</td>
</tr>
</tbody>
</table>
6.4 ANGULAR VARIABLES AND TANGENTIAL VARIABLES

Relations between angular and linear quantities

<table>
<thead>
<tr>
<th>Linear quantities</th>
<th>Angular quantities</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_t$</td>
<td>$\omega$</td>
<td>$v_t = r\omega$ (tangential velocity)</td>
</tr>
<tr>
<td>$a_t$</td>
<td>$\alpha$</td>
<td>$a_t = r\alpha$ (tangential accel.)</td>
</tr>
<tr>
<td>$s$</td>
<td>$\theta$</td>
<td>$s = r\theta$</td>
</tr>
</tbody>
</table>

EXAMPLE 6.6:
A floppy disc in a computer rotates from rest up to an angular speed of 31.4 rad/s in a time of 0.892 s.

a) What is the angular acceleration of the disk, assuming the angular acceleration is uniform?
b) How many rotations does the disk make while coming up to speed?
c) If the radius of the disk is 4.45cm, find the final linear speed of a microbe riding on the rim of the disk.
d) What is the magnitude of the tangential acceleration of the microbe at this time?

Solutions:

a) By using $\omega = \omega_0 + \alpha t$
$$\alpha = \frac{\omega}{t} = \frac{31.4 \text{rad/s}}{0.892 \text{s}} = 35.2 \text{rad/s}^2$$
b) By using \[ \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \]

\[ \theta = \frac{1}{2} (35.2 \text{ rad} / s^2)(0.892 s)^2 = 14.0 \text{ rad} \]

Because \( 2\pi \text{ rad} = 1 \text{ rev} \), this angular displacement corresponds to \( 2.23 \text{ rev} \).

c) \[ v_t = r \omega = (0.0445 \text{ m})(31.4 \text{ rad/s}) = 1.40 \text{ m/s} \]

d) \[ a_t = r \alpha = (0.0445 \text{ m})(35.2 \text{ rad/s}^2) = 1.57 \text{ m/s}^2 \]

**EXAMPLE 6.7**
A 70cm diameter wheel accelerates uniformly about its center from 130rpm to 280rpm in 4.0s. Determine

a) its angular acceleration

b) the radial and tangential component of the linear acceleration of a point on the edge of the wheel 2.0s after it has started accelerating.

**Solutions:**

Convert the rpm values to angular velocities.

\[ \omega_b = \left(130 \text{ rpm} \right) \left( \frac{2\pi \text{ rad}}{1 \text{ rev}} \right) \left( \frac{1 \text{ min}}{60 \text{ sec}} \right) = 13.6 \text{ rad/s} \]

\[ \omega = \left(280 \text{ rpm} \right) \left( \frac{2\pi \text{ rad}}{1 \text{ rev}} \right) \left( \frac{1 \text{ min}}{60 \text{ sec}} \right) = 29.3 \text{ rad/s} \]

(a) The angular acceleration is given by;

\[ \omega = \omega_b + \alpha t \rightarrow \alpha = \frac{\omega - \omega_b}{t} = \frac{29.3 \text{ rad/s} - 13.6 \text{ rad/s}}{4.0 \text{ s}} = 3.93 \text{ rad/s}^2 \approx \frac{3.9 \text{ rad/s}^2}{2} \]

(b) To find the components of the acceleration, the instantaneous angular velocity is needed.

\[ \omega = \omega_b + \alpha t = 13.6 \text{ rad/s} + \left(3.93 \text{ rad/s}^2 \right)(2.0 \text{ s}) = 21.5 \text{ rad/s} \]

The instantaneous radial acceleration is given by \( a_r = \omega^2 r \).

\[ a_r = \omega^2 r = \left(21.5 \text{ rad/s} \right)^2 (0.35 \text{ m}) = 1.6 \times 10^2 \text{ m/s}^2 \]

The tangential acceleration is given by \( a_t = \alpha r \).

\[ a_t = \alpha r = \left(3.93 \text{ rad/s}^2 \right)(0.35 \text{ m}) = 1.4 \text{ m/s}^2 \]
6.5 CENTRIPETAL ACCELERATION AND TANGENTIAL ACCELERATION

Definition of average acceleration:
\[ a = \frac{v_f - v_i}{t_f - t_i} \]
Average acceleration depends on the change in the velocity vector i.e.:
1. magnitude of the velocity
2. direction of the velocity
The change in the direction of the velocity produced the acceleration \( \rightarrow \) centripetal acceleration
Direction : towards the center of the circle.

1. centripetal acceleration (due to circular motion) and
2. tangential acceleration (due to the change in tangential speed)

When both components of acceleration exist simultaneously,
total acceleration :
\[ a = a_t \hat{r} + a_c \hat{r} \]
\[ a^2 = a_c^2 + a_t^2 \]
\[ a = \sqrt{a_c^2 + a_t^2} \]

<table>
<thead>
<tr>
<th></th>
<th>Uniform circular motion</th>
<th>Non uniform circular motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centripetal acceleration</td>
<td>( \sqrt{ } )</td>
<td>( \sqrt{ } )</td>
</tr>
<tr>
<td>Tangential acceleration</td>
<td>Zero</td>
<td>( \sqrt{ } )</td>
</tr>
</tbody>
</table>

EXAMPLE 6.8:
A test car moves at a constant speed of 10 m/s around a circular road of radius 50m. Find the car's
(a) centripetal acceleration
(b) angular speed
(c) tangential acceleration
(d) total acceleration

Solution:
a)  \[ a_c = \frac{v^2}{r} = \frac{(10m/s)^2}{50m} = 2.0m/s^2 \]
Direction: toward the center of curvature of the road.

b) From \( \nu_t = r\omega \),
\[ \omega = \frac{\nu_t}{r} = \frac{10m/s}{50m} = 0.20 rad/s \]

c) \( a_t = 0 \) (speed of the car remains constant)

d) Total acceleration,
\[ a = \sqrt{a_t^2 + a_c^2} = 2.0m/s^2 \]

6.6 TORQUE.  

Torque = the tendency of a force to rotate a body about some axis.
\[ \tau = Fl \]
\[ l = l \sin \theta \]

\[ \rightarrow \text{perpendicular distance from the axis of rotation to the line of the action of the force.} \]
\[ \rightarrow \text{moment arm or lever arm.} \]
SI unit: m.N

EXAMPLE 6.9:
If the torque required to loosen a nut that is holding a flat tire in place on a car has a magnitude of 40.0N.m, what minimum force must be exerted by the mechanic at the end of a 30.0-cm lug wrench to accomplish the task?
SOLUTION:
In order to exert the minimum force, the force must be applied perpendicular to the wrench. With the pivot at the nut, we have
\[ \tau = Fl \]
\[ 40.0 \text{ N} \cdot \text{m} = F (0.300 \text{ m}) \]
So, \( F = 133 \text{ N} \)

EXAMPLE 6.10:
Calculate the net torque (magnitude and direction) on the beam in Figure P8.5 about
a) an axis through O, perpendicular to the page and
b) an axis through C, perpendicular to the page.

Solution:
(a) We have \( \tau_O = (2)(25) \cos30^\circ - (4)(10)\sin20^\circ = 29.6 \text{ N} \cdot \text{m} \) (CCW).
(b) We have \( \tau_C = (2)(30) \sin45^\circ - (2)(10)\sin20^\circ = 35.6 \text{ N} \cdot \text{m} \) (CW).

6.7 RIGID OBJECTS IN EQUILIBRIUM.
The second condition for equilibrium asserts that if an object is in rotational equilibrium, the net torque acting on it about any axis must be zero. That is
\[ \sum \tau = 0 \]
Conditions for Static equilibrium:

1. The resultant external force must be zero.
\[ \sum F = 0 \quad \text{where} \quad \sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \]

2. The resultant external force must be zero.
\[ \sum \tau = 0 \]

**EXAMPLE 6.11:**
A 500N uniform rectangular sign 4.00 m wide and 3.00 m high is suspended from a horizontal, 6.00- m-long uniform, 100N rod, as indicated in figure below. The left end the rod is supported by a hinge and the right end is supported by a thin cable making a 30.0° angle with the vertical.

a) Find the tension, T, in the cable.
b) Find the horizontal and vertical components of force exerted on the left end of the rod by the hinge.

SOLUTION:
(a) The free body diagram of the horizontal rod is shown to the right.

Summing the torques about the left end of the rod yields
\[ -W_{\text{rod}}(3.00 \text{ m}) - W_{\text{sign}}(4.00 \text{ m}) + (T \sin 60^\circ)(6.00 \text{ m}) = 0, \]
giving \( (0.866T)(6.00 \text{ m}) = (100 \text{ N})(3.00 \text{ m}) + (500 \text{ N})(4.00 \text{ m}), \)
or \( T = 443 \text{ N}. \)

(b) Summing the force components in the horizontal direction yields
\[ F_h - T \cos 60^\circ = 0, \text{ or } F_h = 443 \cos 60^\circ = 222 \text{ N}. \]

Summing force components in the vertical direction yields
\[ F_v + T \sin 60^\circ - W_{\text{rod}} - W_{\text{sign}} = 0, \text{ or } F_v = 216 \text{ N}. \]
CENTER OF GRAVITY

A rigid body is an object/systems of particles in which the interparticle distances are fixed and remain constant. The center of mass of a system is the point at which all the mass of the system may be considered to be concentrated. If \( g \) is constant, then the center of gravity is at the center of mass. How to determine the location of the center of gravity???

\[
X_{cg} = \frac{\sum W_i x_i}{\sum W_i} \quad \text{where} \quad X_{cg} = \frac{W_1 x_1 + W_2 x_2 + \ldots}{W_1 + W_2 + \ldots}
\]

6.8 NEWTON'S SECOND LAW FOR ROTATIONAL MOTION ABOUT A FIXED AXIS.

6.9 RELATIONSHIP BETWEEN TORQUE AND ANGULAR ACCELERATION.

From Newton’s Second Law, \( F = ma \)
\[
F = m r^2 \alpha \quad \text{and} \quad \tau = r F
\]
We have:
\[
\tau = m r^2 \alpha
\]
\[
\tau_1 = m r_1^2 \alpha, \quad \tau_2 = m r_2^2 \alpha, \quad \ldots \ldots\ldots
\]
\[
I = m r^2 = \text{moment of inertia}
\]
\[
\sum \tau = I \alpha
\]

The angular acceleration of an object is proportional to the net torque acting on it.

Moments of inertia for various objects of uniform composition as shown in the table below:

<table>
<thead>
<tr>
<th>Object</th>
<th>Location of axis</th>
<th>Moment of inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Thin hoop, radius ( R )</td>
<td>Through center</td>
<td>( MR^2 )</td>
</tr>
<tr>
<td>(b) Thin hoop, radius ( R ), width ( W )</td>
<td>Through center</td>
<td>( \frac{1}{2} MR^2 + \frac{1}{2} MW^2 )</td>
</tr>
<tr>
<td>(c) Solid cylinder, radius ( R )</td>
<td>Through center</td>
<td>( \frac{1}{2} MR^2 )</td>
</tr>
<tr>
<td>(d) Hollow cylinder, inner radius ( R_1 ), outer radius ( R_2 )</td>
<td>Through center</td>
<td>( \frac{1}{2} M(R_1^2 + R_2^2) )</td>
</tr>
<tr>
<td>(e) Uniform sphere, radius ( R )</td>
<td>Through center</td>
<td>( \frac{3}{2} MR^2 )</td>
</tr>
<tr>
<td>(f) Long uniform rod, length ( L )</td>
<td>Through center</td>
<td>( \frac{1}{2} ML^3 )</td>
</tr>
<tr>
<td>(g) Long uniform rod, length ( L ), width ( W )</td>
<td>Through center</td>
<td>( \frac{1}{2} ML^2 )</td>
</tr>
<tr>
<td>(h) Rectangular thin plate, length ( L ), width ( W )</td>
<td>Through center</td>
<td>( \frac{1}{2} ML(L^2 + W^2) )</td>
</tr>
</tbody>
</table>
**EXAMPLE 6.12**

A cable passes over a pulley. Because of friction, the tension in the cable is not the same on opposite sides of the pulley. The force on one side is 120N, and the force on the other side is 100N. \((r=0.81\text{m}, m=2.1\text{kg})\). Calculate the centripetal acceleration of the pulley.

Solution:

The resultant torque is given by
\[
(120 \text{ N})(0.81 \text{ m}) - (100 \text{ N})(0.81 \text{ m}) = 16.2 \text{ N m}
\]

The moment of inertia is:
\[
I = \frac{1}{2} mr^2 = \frac{1}{2} (2.1 \text{ kg})(0.81 \text{ m})^2 = 0.689 \text{ kg m}^2.
\]

Then, \(\tau = I\alpha\) gives
\[
\alpha = \frac{\tau}{I} = \frac{16.2 \text{ Nm}}{0.689 \text{ kg m}^2} = 24 \text{ rad/s}^2.
\]

*Gr 32

**EXAMPLE 6.13**

Given \(m_1=8.0\text{kg}, m_2=3.0\text{kg}, \theta=30^\circ\), and the radius and mass of the pulley are 0.10m and 0.10kg, respectively.

a) What is the acceleration of the masses? [Neglect friction and the string's mass]

b) The pulley has a constant frictional torque of 0.050m.N when the system is in motion, what is the acceleration? [Hint: isolate the forces. The tension in the strings are different. Why?]

(a) Apply Newton’s second law and note \(a = r\alpha\).

\(m_2\):
\[
T_2 - m_2 g = m_2 a,
\]
Eq. (1)

pulley:
\[
T_1 R - T_2 R = I\alpha = \frac{1}{2} MR^2\alpha = \frac{1}{2} MRa,
\]

or
\[
T_1 - T_2 = 0.5Ma,
\]
Eq. (2)

\(m_1\):
\[
m_1 g \sin\theta - T_1 = m_1 a,
\]
Eq. (3)
Eq. (1) + Eq. (2) + Eq. (3) gives \[ m_1 g \sin \theta - m_2 g = (m_1 + m_2 + 0.5M)a, \]
so \[ a = \frac{(m_1 \sin \theta - m_2)g}{m_1 + m_2 + 0.5M} = \frac{[8.0 \text{ kg} \sin 30^\circ - (3.0 \text{ kg})](9.80 \text{ m/s}^2)}{8.0 \text{ kg} + 3.0 \text{ kg} + 0.5(0.10 \text{ kg})} = 0.89 \text{ m/s}^2. \]

(b) pulley: \[ T_1 R - T_2 R - \tau_f = \frac{1}{2} MR^2 \alpha = \frac{1}{2} MRa, \]
or \[ T_1 - T_2 = \frac{\tau_f}{R} = 0.5Ma. \]

Eq. (2)

\[ a = \frac{(m_1 \sin \theta - m_2)g - \frac{\tau_f}{R}}{m_1 + m_2 + 0.5M} = \]

\[ \frac{[8.0 \text{ kg} \sin 30^\circ - (3.0 \text{ kg})](9.80 \text{ m/s}^2) - 0.050 \text{ m·N}}{0.10 \text{ m}} \]

\[ = \frac{0.84 \text{ m/s}^2}. \]

The tensions are different because of the frictional torque.

6.10 ROTATIONAL WORK AND KINETIC ENERGY.

The rotational work \( W_R \) done by a constant torque:

\[ W_R = \tau \theta \]

A body rotating about some axis with an angular speed \( \omega \) has rotational KE given by \( \frac{1}{2} I \omega^2 \).

\[ KE_r = \sum \left( \frac{1}{2} m v^2 \right) = \sum \left( \frac{1}{2} mr^2 \omega^2 \right) = \frac{1}{2} \left( \sum mr^2 \right) \omega^2 \]

\[ KE_r = \frac{1}{2} I \omega^2 \]

\[ I = \sum mr^2 \rightarrow \text{moment of inertia of the body.} \]

\( (KE_t + KE_r + PE_g)_{I} = (KE_t + KE_r + PE_g)_{f} \)
EXAMPLE 6.14
A person opens a door by applying a 15-Newton force perpendicular to it at a distance 0.90m from the hinges. The door is pushed wide open (to $120^0$) in 2.0s.
a) How much work was done?
b) What was the average power delivered?

Solutions:
(a) $W = \tau \theta = r F \theta = (0.90 \text{ m})(15 \text{ N})(120^0)(\pi \text{ rad/180}^0) = [28 \text{ J}]$.
(b) $P = \frac{W}{\Delta t} = \frac{28 \text{ J}}{2.0 \text{ s}} = [14 \text{ W}]$.

6.11 ANGULAR MOMENTUM. AND 8.12 THE CONSERVATION OF ANGULAR MOMENTUM.

$$\tau = I \alpha = I \left( \frac{\omega - \omega_0}{\Delta t} \right) = \frac{I \omega - I \omega_0}{\Delta t}$$

Definition of angular momentum:

$L = I \omega$

$$\tau = \frac{\Delta L}{\Delta t}$$

SI unit: kg.m$^2$/s

NSL in terms of momentum

$$\tau = \frac{\Delta L}{\Delta t}$$

In the absence of an external, unbalanced torque, the total angular momentum of a system is conserved (remains constant).

If $\sum \tau = 0$

$L_i = L_f$

$I_i \omega_i = I_f \omega_f$

Conservation of angular momentum:

The angular mtm of a system is conserved when the net external torque acting on the system is zero. That is, when $\sum \tau = 0$, the initial angular mtm equals the final angular mtm.
**EXAMPLE 6.15:**

A skater has a moment of inertia of 100 kgm$^2$ when his arms are outstretched and a moment of inertia of 75kgm$^2$ when his arms are tucked in close to his chest. If he starts to spin at an angular speed of 2.0rps with his arm outstretched, what will his angular speed be when they are tucked in?

Solution:

From angular momentum conservation: \( I \omega = I_o \omega_o \).

\[
\omega = \frac{I_o \omega_o}{I} = \frac{(100 \text{ kg} \cdot \text{m}^2)(2.0 \text{ rps})}{75 \text{ kg} \cdot \text{m}^2} = \boxed{2.7 \text{ rps}}.
\]