Laplace Transform
Course Objectives

To provide an analysis skill of the continuous-time signals and systems as reflected to their roles in engineering practice.

To expose students to both the time-domain and frequency-domain methods of analyzing signals and systems.

To illustrate the potential applications of this course as a Pre-requisite course to communication engineering and principles, digital signal processing and control system.
After completion of this course the students will be able to:

Classify, characterize and conduct basic of signals and systems.

Analyze continuous-time signals and systems in time domain using convolution.

Analyze continuous-time signals and systems in frequency domain using Laplace transform.

Analyze continuous-time signals and systems in frequency domain using Fourier series and Fourier transform.

Acquire introductory-level knowledge of discrete-time signals and systems, and sampling theory.

Work in group to perform basic simulation of signals and systems analysis.
Laplace Transform

- The Laplace Transform
  - Finding the Inverse Transform
  - Pole-Zero
  - Region of Convergence (ROC)
- Some Properties of the Laplace Transform
- Solution of Differential and Integro-Differential Equations
  - Zero-State Response
  - Stability
  - Inverse System
- Block Diagrams
- Bilateral Laplace Transform
- Filter Design by Placement of Poles and Zeros of H(s)
Why frequency domain?

- We’ve been discussing about systems strictly in the time-domain, all signals were represented as functions of time.
- It is often much easier to analyze signals and systems when they are represented in the frequency domain.
- Subject of Signals and Systems consists primarily on the following concepts:
  1. Writing signals as functions of frequency
  2. Looking at how systems respond to inputs of different frequencies
  3. Developing tools for switching between time-domain and frequency-domain representations
  4. Learning how to determine which domain is best suited for particular problem
Complex frequency

- **Frequency** = the number of cycles per second of a period signal.

- *(this is correct definition, but now is the time to broaden your horizon)*

- For complete generality, we will allow this frequency to be a complex number.

- Here we declare that any function written in the form of $K e^{s t}$ (where $K$ and $s$ are in general complex constant) is characterized by the complex frequency $s$, which can be expanded as $s = \sigma + j \omega$. 
Why Laplace Transform?

- Laplace transform is the **dual** (or complement) of the time-domain analysis.
- In the **time-domain** analysis, we break the input $x(t)$ into **impulsive component**, and sum the system response to all these components.
- In the **frequency-domain** analysis, we break the input $x(t)$ into **exponentials components** of the form $e^{st}$, where $s$ is the complex frequency:
  \[ s = \alpha + j\omega \]
- Laplace transform is the tool to map signals and system behavior from the time-domain into the frequency domain.
The Laplace Transform

6.1 Introduction

♦ Basic concepts:
1. The Laplace transform (LT) provides a broader characterization of continuous-time LTI systems and their interaction with signals than is possible with Fourier transform.

   Signal that is not absolutely integral \[ \xrightarrow{\text{Laplace transform}} \]

   \[ \xrightarrow{\text{Fourier transform}} \]

2. Continuous-time complex exponentials are eigenfunctions of LTI systems.

   The convolution of time signals becomes multiplication of the associated Laplace transform.

3. Two varieties of LT: (1) unilateral, or one-sided, and (2) bilateral, or two-sided.

   ♦ The bilateral Laplace transform (BLT) offers insight into the nature of system characteristics such as stability, causality, and frequency response.
6.2 The Laplace Transform

- Complex exponential with complex frequency $s = \sigma + j\omega$:

\[ e^{st} = e^{\sigma t} \cos(\omega t) + j e^{\sigma t} \sin(\omega t). \]  

(6.1)  

Figure 6.1 (p. 483)  
Real and imaginary parts of the complex exponential $e^{st}$, where $s = \sigma + j\omega$.  

$p \equiv \pi; \quad \nu \equiv \omega; \quad s \equiv \sigma$
6.2.1 Eigenfunction Property of $e^{st}$

1. LTI system: 

\[ x(t) = e^{st} \]

LTI system, $h(t)$

\[ y(t) = x(t) * h(t) \]

$h(t) \equiv$ impulse response

2. System output:

\[ y(t) = H \{ x(t) \} \]

\[ = h(t) * x(t) \]

\[ = \int_{-\infty}^{\infty} h(\tau)x(t-\tau) \, d\tau \]

3. Transfer function:

\[ H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} \, d\tau \quad (6.2) \]

\[ y(t) = H \{ e^{st} \} = H(s)e^{st} \]

$e^{st} \equiv$ Eigenfunction; $H(s) \equiv$ eigenvalue

The action of the LTI system on an input $e^{st}$ is multiplication by the transfer function $H(s)$. 
4. Polar form of complex-valued transfer function:

\[ H(s) = |H(s)| e^{j\phi(s)} \]

\[ y(t) = |H(s)| e^{j\phi(s)} e^{st} \]

We use \( s = \sigma + j\omega \) to obtain

\[
y(t) = |H(\sigma + j\omega)| e^{\sigma t} e^{j\omega t + \phi(\sigma + j\omega)} \\
= |H(\sigma + j\omega)| e^{\sigma t} \cos(\omega t + \phi(\sigma + j\omega)) + j |H(\sigma + j\omega)| e^{\sigma t} \sin(\omega t + \phi(\sigma + j\omega))
\]

The LTI system changes the amplitude of the input by \( |H(\sigma + j\omega)| \) and shifts the phase of the sinusoidal components by \( \phi(\sigma + j\omega) \).

6.2.2 Laplace Transform Representation

1. A representation of arbitrary signals as a weighted superposition of eigenfunctions \( e^{st} \):

Substituting \( s = \sigma + j\omega \) into Eq.(2) and using \( t \) as the variable of integration, we obtain

\[
H(\sigma + j\omega) = \int_{-\infty}^{\infty} h(t) e^{-(\sigma + j\omega)t} dt \\
= \int_{-\infty}^{\infty} \left[ h(t) e^{-\sigma t} \right] e^{-j\omega t} dt
\]
The Laplace Transform

1. **$H(\sigma + j\omega)$ is the Fourier transform of $h(t)e^{-\sigma t}$.**

2. **Inverse Fourier transform of $H(\sigma + j\omega)$:**

   
   $$h(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)e^{j\omega t} \, d\omega$$

   
   $$h(t) = e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)e^{j\omega t} \, d\omega$$

   
   $$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)e^{(\sigma+j\omega)t} \, d\omega$$

   
   (6.3)

   Substituting $s = \sigma + j\omega$ and $d\omega = ds/j$ into Eq.(6.3) we get

   $$h(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} H(s)e^{st} \, ds$$

   (6.4)

   The limits on the integral are also a result of the substitution $s = \sigma + j\omega$

3. **Laplace transform of $x(t)$:**

   $$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt$$

   (6.5)
4. Inverse Laplace transform of $X(s)$:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$  \hspace{1cm} (6.6)

- Notation:

$$x(t) \leftrightarrow \mathcal{L} \rightarrow X(s)$$

6.2.3 Convergence

1. A necessary condition for convergence of the Laplace transform is the absolute integrability of $x(t)e^{-\sigma t}$:

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty$$

2. The range of $\sigma$ for which the Laplace transform converges is termed the region of convergence (ROC).

Example:

Fourier transform of $x(t) = e^{t}u(t)$ does not exist, but if $\sigma > 1$, $x(t)e^{-\sigma t} = e^{(1-\sigma)t}u(t)$ is absolutely integrable, and so the Laplace transform, which is the Fourier transform of $x(t)e^{-\sigma t}$, does exist.

Fig. 6.2.
The Laplace Transform

Figure 6.2 (p. 485)
The Laplace transform applies to more general signals than the Fourier transform does. (a) Signal for which the Fourier transform does not exist. (b) Attenuating factor associated with Laplace transform. (c) The modified signal $x(t)e^{-\sigma t}$ is absolutely integrable for $\sigma > 1$.

6.2.4 The s-Plane

1. Complex $s$-plane: Fig. 6.3.
2. Complex frequency: $s = \sigma + j\omega$
3. Relation between FT and LT:

   $$X(j\omega) = X(s)\bigg|_{\sigma=0}$$  \hfill (6.7)

4. The $j\omega$-axis divides the $s$-plane in half: left-half and right-half $s$-plane.

Horizontal axis of $s$-plane = real part of $s$; vertical axis of $s$-plane = imaginary part of $s$. 
Figure 6.3 (p. 486)
The s-plane. The horizontal axis is Re\{s\} and the vertical axis is Im\{s\}. Zeros are depicted at \(s = -1\) and \(s = -4 \pm 2j\), and poles are depicted at \(s = -3\), \(s = 2 \pm 3j\), and \(s = 4\).

6.2.5 Poles and Zeros

1. Laplace transform \(X(s)\) in terms of a ratio of two polynomials:

\[
X(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}
\]

\[
X(s) = \frac{b_M \prod_{k=1}^{M} (s - c_k)}{\prod_{k=1}^{N} (s - d_k)}
\]

\(c_k\) = zeros of \(X(s)\); \(d_k\) = poles of \(X(s)\)

Fig. 6.3: \(\times\) \equiv pole; \(\circ\) \equiv zero
Exponential Function $e^{st}$ (1)

- Exponential function is very important in signals & system, and the parameter $s$ is a complex variable given by:
  \[ s = \sigma + j\omega \]

- Therefore
  \[ e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} (\cos \omega t + j \sin \omega t) \]

- Since the conjugate of $s$
  \[ s^* = \sigma - j\omega \]

- Then
  \[ e^{s^*t} = e^{(\sigma-j\omega)t} = e^{\sigma t} e^{-j\omega t} = e^{\sigma t} (\cos \omega t - j \sin \omega t) \]

- and
  \[ e^{\sigma t} \cos \omega t = \frac{1}{2} (e^{st} + e^{s^*t}) \]
Exponential Function $e^{st}$ (2)

- If $\sigma=0$, then we have the function $e^{j\omega t}$, which has a real frequency of $\omega$.
- Therefore the complex variable $s = \sigma + j\omega$ is the **complex frequency**.
- The function $e^{st}$ can be used to describe a very large class of signals and function. Here are a number of examples:
  1. A constant $k = ke^{0t}$ \hspace{1cm} (s = 0)
  2. A monotonic exponential $e^{\sigma t}$ \hspace{1cm} (s = $\sigma$, $\omega = 0$)
  3. A sinusoid $\cos \omega t$ \hspace{1cm} (s = $\pm j\omega$, $\sigma = 0$)
  4. An exponentially varying sinusoid $e^{\sigma t} \cos \omega t$ \hspace{1cm} (s = $\sigma \pm j\omega$)
Exponential Function $e^{st}$ (3)
The Complex Frequency Plane $s = \sigma + j\omega$

$s$ on left of $y$-axis  
$s$ on $y$-axis  
$s$ on right of $y$-axis

$s$ on $x$-axis

$s$ plane
The form $e^{st}$ are Eigenfunctions of CT

- Why complex exponentials of the form $e^{st}$ are indeed eigenfunctions of continuous-time systems.

- CT convolution formula

- Substitute in $x(t) = e^{st}$ input

- Move $e^{st}$ outside the integral

- Use definition of Laplace transform
Function of the form $e^{st}$ are indeed eigenfunctions of continuous-time systems. It means that when such a function is an input to a CT LTI system, the output is a function of the exact same complex frequency as the input, except it is multiplied by a scaling factor. The coefficient $H(s)$ is in general a complex number.
Definition of **Two-sided (Bilateral) Laplace Transform**

- For a signal $x(t)$, its **Laplace transform** is defined by:
  \[
  X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt
  \]

- The signal $x(t)$ is said to be the **inverse Laplace transform** of $X(s)$. It can be shown that
  \[
  x(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} X(s)e^{st} ds
  \]

- where $C$ is a constant chosen to ensure the **convergence** of the first integral.
- This general definition is known as **two-sided** (or **bilateral**) Laplace Transform.
Definition of **One-sided (Unilateral)** Laplace Transform

- Symbolically, it can be shown as $X(s) = \mathcal{L} \{x(t)\}$ and $x(t) = \mathcal{L}^{-1} \{X(s)\}$.
- It is also a common practice to use a bidirectional arrow to indicate a Laplace transform pair $x(t) \leftrightarrow X(s)$.
- Note that:
  - $\mathcal{L}^{-1} \{\mathcal{L} \{x(t)\}\} = x(t)$ and $\mathcal{L} \{\mathcal{L}^{-1} \{X(s)\}\} = X(s)$.
- If all the signals are causal its Laplace transform is defined by:

$$X(s) = \int_{0}^{\infty} x(t)e^{-st} \, dt$$

**One-sided (unilateral) vs. Two-sided (bilateral):**

- Unilateral is used in the solution of differential equation with initial condition and analysis of causal LTI CT systems.
- Bilateral is used in the steady-state (zero initial condition) analysis of LTI CT system and non-causal system.
Linearity of the Laplace Transform

- The principle of **superposition** holds in Laplace transform

\[
\mathcal{L}[a_1 x_1(t) + a_2 x_2(t)] = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-st} dt
\]

\[
= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-st} dt
\]

\[
= a_1 X_1(s) + a_2 X_2(s)
\]
Unilateral Laplace Transform Pairs (1)

- Finding inverse Laplace transform requires integration in the complex plane – beyond scope of this course.
- So use a Laplace transform table (analogous to the convolution table)

<table>
<thead>
<tr>
<th>No.</th>
<th>$x(t)$</th>
<th>$X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>3</td>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>4</td>
<td>$t^n u(t)$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>No.</td>
<td>( x(t) )</td>
<td>( X(s) )</td>
</tr>
<tr>
<td>-----</td>
<td>-------------</td>
<td>-------------</td>
</tr>
<tr>
<td>5</td>
<td>( e^{\lambda t} u(t) )</td>
<td>( \frac{1}{s - \lambda} )</td>
</tr>
<tr>
<td>6</td>
<td>( te^{\lambda t} u(t) )</td>
<td>( \frac{1}{(s - \lambda)^2} )</td>
</tr>
<tr>
<td>7</td>
<td>( t^n e^{\lambda t} u(t) )</td>
<td>( \frac{n!}{(s - \lambda)^{n+1}} )</td>
</tr>
<tr>
<td>8a</td>
<td>( \cos bt u(t) )</td>
<td>( \frac{s}{s^2 + b^2} )</td>
</tr>
<tr>
<td>8b</td>
<td>( \sin bt u(t) )</td>
<td>( \frac{b}{s^2 + b^2} )</td>
</tr>
<tr>
<td>9a</td>
<td>( e^{-at} \cos bt u(t) )</td>
<td>( \frac{s + a}{(s + a)^2 + b^2} )</td>
</tr>
<tr>
<td>9b</td>
<td>( e^{-at} \sin bt u(t) )</td>
<td>( \frac{b}{(s + a)^2 + b^2} )</td>
</tr>
</tbody>
</table>
### Unilateral Laplace Transform Pairs (3)

<table>
<thead>
<tr>
<th>No.</th>
<th>$x(t)$</th>
<th>$X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10a</td>
<td>$re^{-at} \cos( bt + \theta ) u(t)$</td>
<td>( \frac{(r \cos \theta)s + (ar \cos \theta - br \sin \theta)}{s^2 + 2as + (a^2 + b^2)} )</td>
</tr>
<tr>
<td>10b</td>
<td>$re^{-at} \cos( bt + \theta ) u(t)$</td>
<td>( \frac{0.5re^{j\theta}}{s + a - jb} + \frac{0.5re^{-j\theta}}{s + a + jb} )</td>
</tr>
<tr>
<td>10c</td>
<td>$re^{-at} \cos( bt + \theta ) u(t)$</td>
<td>( \frac{As + B}{s^2 + 2as + c} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c - a^2}}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta = \tan^{-1}\left(\frac{Aa - B}{A\sqrt{c - a^2}}\right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b = \sqrt{c - a^2}$</td>
<td></td>
</tr>
<tr>
<td>10d</td>
<td>$e^{-at} \left[A \cos bt + \frac{B - Aa}{b} \sin bt\right] u(t)$</td>
<td>( \frac{As + B}{s^2 + 2as + c} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b = \sqrt{c - a^2}$</td>
<td></td>
</tr>
</tbody>
</table>
Region of Convergence (ROC)

- Also called the region of existence.
- The region of convergence (ROC) of a Laplace transform is defined as the set of values of $s$ for which the Laplace transform integral can be evaluated.
- There are several rules that can be used to quickly determine the ROC without performing any integration. All that is needed is the type of signals and the location of the poles of the transform.
- A pole is defined as a value of $s$ that causes the denominator of the transform to become zero.
Types of Signals

- **left-sided signal**
  (starts somewhere, ends at $-\infty$)

- **right-sided signal**
  (starts somewhere, ends at $+\infty$)

- **two-sided signal**
  (starts at $-\infty$, ends at $+\infty$)

- **finite-duration signal**
  (starts somewhere, ends somewhere)
Rules of the ROC

- The ROC is always a region of the s-plane to the left or right of a vertical line, or a strip between two vertical line.
- The ROC never contains any poles.
- If $x(t)$ is right-sided, then the ROC is right-sided. i.e. $\text{Re}\{s\}>a$, where $a$ is the Re{rightmost pole}
- If $x(t)$ is left-sided, then the ROC is left-sided. i.e. $\text{Re}\{s\}<a$, where $a$ is the Re[leftmost pole]
- If $x(t)$ is two-sided or the sum of a left and right sided signal, then the ROC is either a strip $(a<\text{Re}\{s\}<b)$or else the individual ROC will not overlap, producing the null set.
- If $x(t)$ is finite duration, then the ROC is the entire s-plane.
For right-sided signal, ROC is always to the right of s plane.

ROC, \( \sigma > \alpha \)

Right-sided signal
(Starts somewhere, ends at \(+\infty\)
For left-sided signal, ROC is always to the left of s plane.

\( ROC, \sigma < -\alpha \)

left-sided signal

(starts somewhere, ends at \(-\infty\))
For two sided signal, ROC is two-sided.
Inverse Laplace Transform

- In principle we can recover \( x(t) \) from \( X(s) \) via

\[
x(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} X(s)e^{st}ds
\]

- Rather to perform the inverse Laplace transform, we will merely manipulate the given expression until we see patterns we recognize from the Laplace transform table.

- This is basically a heuristic scheme and is one that will become more obvious with practice.

- PFE (partial fraction expansion) is very helpful for this purpose.
## Properties of the Laplace Transform

- Time Shifting Property
- Frequency Shifting Property
- Time-Differentiation Property
- Time-Integration Property
- Time-Convolution & Frequency Convolution Property
- Initial-Final Value
<table>
<thead>
<tr>
<th>Operation</th>
<th>( x(t) )</th>
<th>( X(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>( x_1(t) + x_2(t) )</td>
<td>( X_1(s) + X_2(s) )</td>
</tr>
<tr>
<td>Scalar multiplication</td>
<td>( kx(t) )</td>
<td>( kX(s) )</td>
</tr>
<tr>
<td>Time differentiation</td>
<td>( \frac{dx}{dt} )</td>
<td>( sX(s) - x(0^-) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{d^2x}{dt^2} )</td>
<td>( s^2X(s) - sx(0^-) - \dot{x}(0^-) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{d^3x}{dt^3} )</td>
<td>( s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{d^n x}{dt^n} )</td>
<td>( s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-) )</td>
</tr>
<tr>
<td>Time integration</td>
<td>( \int_{0^-}^{t} x(\tau) , d\tau )</td>
<td>( \frac{1}{s}X(s) )</td>
</tr>
<tr>
<td></td>
<td>( \int_{-\infty}^{t} x(\tau) , d\tau )</td>
<td>( \frac{1}{s}X(s) + \frac{1}{s} \int_{-\infty}^{0^-} x(t) , dt )</td>
</tr>
<tr>
<td>Operation</td>
<td>$x(t)$</td>
<td>$X(s)$</td>
</tr>
<tr>
<td>---------------------------</td>
<td>-------------------------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td>Time shifting</td>
<td>$x(t - t_0)u(t - t_0)$</td>
<td>$X(s)e^{-st_0}$ $t_0 \geq 0$</td>
</tr>
<tr>
<td>Frequency shifting</td>
<td>$x(t)e^{st_0}$</td>
<td>$X(s - s_0)$</td>
</tr>
<tr>
<td>Frequency differentiation</td>
<td>$-tx(t)$</td>
<td>$\frac{dX(s)}{ds}$</td>
</tr>
<tr>
<td>Frequency integration</td>
<td>$\frac{x(t)}{t}$</td>
<td>$\int_s^\infty X(z) , dz$</td>
</tr>
<tr>
<td>Scaling</td>
<td>$x(at), a \geq 0$</td>
<td>$\frac{1}{a} X\left(\frac{s}{a}\right)$</td>
</tr>
<tr>
<td>Time convolution</td>
<td>$x_1(t) * x_2(t)$</td>
<td>$X_1(s)X_2(s)$</td>
</tr>
<tr>
<td>Frequency convolution</td>
<td>$x_1(t)x_2(t)$</td>
<td>$\frac{1}{2\pi j} X_1(s) * X_2(s)$</td>
</tr>
<tr>
<td>Initial value</td>
<td>$x(0^+)$</td>
<td>$\lim_{s \to \infty} sX(s)$ $(n &gt; m)$</td>
</tr>
<tr>
<td>Final value</td>
<td>$x(\infty)$</td>
<td>$\lim_{s \to 0} sX(s)$ [poles of $sX(s)$ in LHP]</td>
</tr>
</tbody>
</table>
Linearity Property

- Linearity

\[ \alpha x(t) + \beta y(t) \overset{L}{\leftrightarrow} \alpha X(s) + \beta Y(s) \]
Time-Shifting Property

\[ x(t - t_0) \overset{L}{\leftrightarrow} X(s) e^{-st_0} \quad t_0 > 0 \]

- Delaying \( x(t) \) by \( t_0 \) (i.e. time shifting) amount to multiply its transform \( X(s) \) by \( e^{-st_0} \).
- The more accurate statement of the time-shifting property is

\[ x(t) u(t) \overset{L}{\leftrightarrow} X(s) \]

\[ x(t - t_0) u(t - t_0) \overset{L}{\leftrightarrow} X(s) e^{-st_0} \quad t_0 > 0 \]
Frequency-Shifting Property

\[ x(t)e^{s_0t} \xleftrightarrow{L} X(s - s_0) \]

- Frequency shifting transform \( X(s) \) by \( s_0 \) amounts to multiplying its time signal \( x(t) \) by \( e^{s_0t} \).

- Observe symmetry (or duality) between frequency-shift and time-shift properties.

\[ x(t - t_0) \xleftrightarrow{L} X(s)e^{-st_0} \quad t_0 > 0 \]
Time Scaling – Frequency Scaling

- **Time scaling**

\[ x(at) \overset{L}{\leftrightarrow} \frac{1}{a} \left[ X\left(\frac{s}{a}\right) \right] \quad a > 0 \]

- **Frequency scaling**

\[ \frac{1}{a} x\left(\frac{t}{a}\right) \overset{L}{\leftrightarrow} X(as) \quad a > 0 \]
Time-differentiation Property

• First order time differentiation

\[ \frac{d}{dt} x(t) \overset{L}{\leftrightarrow} sX(s) - x(0^-) \]

• Second order time differentiation

\[ \frac{d^2}{dt^2} x(t) \overset{L}{\leftrightarrow} s^2 X(s) - sx(0^-) - \frac{d}{dt} x(0^-) \]
Frequency-differentiation Property

- Frequency differentiation property

\[- tx(t) \overset{L}{\longleftrightarrow} \frac{d}{ds} X(s)\]
Time-integration Property

**Frequency-integration Property**

- Time-integration Property
  \[
  \int_{0}^{t} x(\tau) d\tau \xleftrightarrow{L} \frac{X(s)}{s}
  \]

- Frequency-integration Property
  \[
  \frac{x(t)}{t} \xleftrightarrow{L} \int_{s}^{t} X(z) dz
  \]
Time Convolution & Frequency Convolution Property

- Time-convolution property
  \[ x(t) \ast h(t) \leftrightarrow X(s)H(s) \]

- Convolution in time domain is equivalent to multiplication in s (frequency) domain

- Frequency-convolution property
  \[ x(t)h(t) \leftrightarrow \frac{1}{2\pi j} X(s) \ast H(s) \]

- Convolution in frequency s domain is equivalent to multiplication in time domain
Initial and Final Value Theorem

How to find the initial and final value of a function $x(t)$ if we know its Laplace Transform $X(s)$?

**Initial Value Theorem**

$$\lim_{t \to 0} x(t) = x(0^+) = \lim_{s \to \infty} sX(s)$$

**Final Value Theorem**

$$\lim_{t \to \infty} x(t) = x(\infty) = \lim_{s \to 0} sX(s)$$

**Conditions:**
- Laplace transforms of $x(t)$ and $dx/dt$ exist.
- $X(s)$ numerator power ($M$) is less than denominator power ($N$), i.e. $M < N$.
- $sX(s)$ poles are all on the LFP or origin.
Introduction to the Laplace Transform
Definition of the Laplace Transform

One-sided LT: \[ \mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st} \, dt \]

where \( s = \sigma + j\omega \)

- The Laplace transform is an integral transformation of a function \( f(t) \) from the time domain into the complex frequency domain, given \( F(s) \).
Convergence Region:
\[ \int_{0^-}^{\infty} e^{-\sigma t} |f(t)| \, dt < \infty \quad \text{Re}(s) = \sigma > \sigma_c \]

Inverse LT: \( \mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st} \, ds \)

LT Pair: \( f(t) \Leftrightarrow F(s) \)
Region of Convergence
Example

- Determine the Laplace transform of each of the following functions:
  - (a) $u(t)$,
  - (b) $e^{-at}u(t), \ a \geq 0$,
  - (c) $\delta(t)$.

Solution:

(a)
Example

\[
\mathcal{L}[u(t)] = \int_{0}^{\infty} 1 e^{-st} \, dt = \frac{1}{s} e^{-st} \bigg|_{0}^{\infty} \\
= -\frac{1}{s} (0) + \frac{1}{s} (1) = \frac{1}{s}
\]
Example

\[ \mathcal{L}[e^{-at} u(t)] = \int_0^\infty e^{-st} e^{-at} dt \]

\[ = -\frac{1}{s + a} e^{-(s+a)t} \bigg|_0^\infty = \frac{1}{s + a} \]
\[ L[\delta(t)] = \int_0^{\infty} \delta(t)e^{-st} \, dt = e^{-0} = 1 \]
Example

- Determine the Laplace transform of function: \( \sin \omega t \ u(t) \).

- Solution:

\[
F(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t)e^{-st} \, dt = \int_0^\infty \left( \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right)e^{-st} \, dt
\]

\[
= \frac{1}{2j} \int_0^\infty \left( e^{-(s-j\omega)t} - e^{-(s+j\omega)t} \right) dt
\]

\[
= \frac{1}{2j} \left( \frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right) = \frac{\omega}{s^2 + \omega^2}
\]
Properties of the Laplace Transform

- **Linearity**
  \[
  \mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 f_1(s) + a_2 f_2(s)
  \]

- **Scaling**
  \[
  \mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)
  \]

- **Time shift**
  \[
  \mathcal{L}[f(t-a)u(t-a)] = e^{-sa} F(s)
  \]
- Frequency Shift

\[ \mathcal{L}[e^{-at}f(t)] = F(s + a) \]
• Time Differentiation

\[ \mathcal{L}[f'(t)] = sF(s) - f(0^-) \]
\[ \mathcal{L}[f''(t)] = s^2 F(s) - sf(0^-) - f'(0^-) \]
\[ \mathcal{L}\left[ \frac{d^n f}{dt^n} \right] = s^n F(s) - s^{n-1} f(0^-) \]
\[ - s^{n-2} f'(0^-) - \cdots - s^0 f^{(n-1)}(0^-) \]
• Time Integration

\[ \mathcal{L} \left[ \int_0^t f(t) \, dt \right] = \frac{1}{s} F(s) \]

• Frequency Differentiation

\[ \mathcal{L}[tf(t)] = -\frac{dF(s)}{ds} \]

\[ \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n} \]
• Time Periodicity

\[ F(s) = \frac{F_1(s)}{1 - e^{-Ts}} \]

• Initial and Final Values

\[ f(0) = \lim_{s \to \infty} sF(s) \]

\[ f(\infty) = \lim_{s \to 0} sF(s) \]
**TABLE 15.1** Properties of the Laplace transform.

<table>
<thead>
<tr>
<th>Property</th>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( a_1 f_1(t) + a_2 f_2(t) )</td>
<td>( a_1 F_1(s) + a_2 F_2(s) )</td>
</tr>
<tr>
<td>Scaling</td>
<td>( f(at) )</td>
<td>( \frac{1}{a} F(\frac{s}{a}) )</td>
</tr>
<tr>
<td>Time shift</td>
<td>( f(t - a)u(t - a) )</td>
<td>( e^{-as}F(s) )</td>
</tr>
<tr>
<td>Frequency shift</td>
<td>( e^{-at}f(t) )</td>
<td>( F(s + a) )</td>
</tr>
<tr>
<td>Time differentiation</td>
<td>( \frac{df}{dt} )</td>
<td>( sF(s) - f(0^-) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{d^2f}{dt^2} )</td>
<td>( s^2 F(s) - sf(0^-) - f'(0^-) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{d^3f}{dt^3} )</td>
<td>( s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-) )</td>
</tr>
<tr>
<td></td>
<td>( \frac{d^nf}{dt^n} )</td>
<td>( s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-) )</td>
</tr>
<tr>
<td>Time integration</td>
<td>( \int_0^t f(t) dt )</td>
<td>( \frac{1}{s} F(s) )</td>
</tr>
<tr>
<td>Frequency differentiation</td>
<td>( tf(t) )</td>
<td>( -\frac{d}{ds} F(s) )</td>
</tr>
<tr>
<td>Frequency integration</td>
<td>( \frac{f(t)}{t} )</td>
<td>( \int_s^\infty F(s) ds )</td>
</tr>
<tr>
<td>Time periodicity</td>
<td>( f(t) = f(t + nT) )</td>
<td>( \frac{F_1(s)}{1 - e^{-sT}} )</td>
</tr>
<tr>
<td>Initial value</td>
<td>( f(0) )</td>
<td>( \lim_{s \to \infty} s F(s) )</td>
</tr>
<tr>
<td>Final value</td>
<td>( f(\infty) )</td>
<td>( \lim_{s \to 0} s F(s) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( f_1(t) * f_2(t) )</td>
<td>( F_1(s) F_2(s) )</td>
</tr>
</tbody>
</table>

**TABLE 15.2** Laplace transform pairs.*

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(t) )</td>
<td>1</td>
</tr>
<tr>
<td>( u(t) )</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( \frac{1}{s + a} )</td>
</tr>
<tr>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>( i^n )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
</tr>
<tr>
<td>( te^{-at} )</td>
<td>( \frac{1}{(s + a)^2} )</td>
</tr>
<tr>
<td>( i^n e^{-at} )</td>
<td>( \frac{n!}{(s + a)^{n+1}} )</td>
</tr>
<tr>
<td>( \sin \omega t )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( \cos \omega t )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( \sin(\omega t + \theta) )</td>
<td>( \frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( \cos(\omega t + \theta) )</td>
<td>( \frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-at} \sin \omega t )</td>
<td>( \frac{\omega}{(s + a)^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-at} \cos \omega t )</td>
<td>( \frac{s + a}{(s + a)^2 + \omega^2} )</td>
</tr>
</tbody>
</table>

*Defined for \( t \geq 0 \); \( f(t) = 0 \), for \( t < 0 \).
Example

- Obtain the Laplace transform of \( f(t) = \delta(t) + 2u(t) - 3e^{-2t}, \quad t \geq 0. \)
- **Solution:**

\[
F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}u(t)]
\]

\[
= 1 + 2 \frac{1}{s} - 3 \frac{1}{s + 2} = \frac{s^2 + s + 4}{s(s + 2)}
\]
Example

- Determine the Laplace transform of \( f(t) = t^2 \sin 2t \ u(t) \).
- Solution:

\[
\mathcal{L}[\sin 2t] = \frac{2}{s^2 + 2^2}
\]

\[
F(s) = \mathcal{L}[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2} \left( \frac{2}{s^2 + 4} \right)
\]

\[
= \frac{d}{ds} \left( \frac{-4s}{(s^2 + 4)^2} \right) = \frac{12s^2 - 16}{(s^2 + 4)^3}
\]
Example

- Find the Laplace transform of the gate function
Example

\[ g(t) = 10[u(t - 2) - u(t - 3)] \]

\[ G(s) = 10 \left( \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} \right) = \frac{10}{s} (e^{-2s} - e^{-3s}) \]
Example

- Find the Laplace transform of the periodic function in this figure.
Example

\[ f_1(t) = 2t[u(t) - u(t - 1)] = 2tu(t) - 2tu(t - 1) \]
\[ = 2tu(t) - 2(t - 1 + 1)u(t - 1) \]
\[ = 2tu(t) - 2(t - 1)u(t - 1) - 2u(t - 1) \]

\[ F(s) = \frac{2}{s^2} - 2 \frac{e^{-s}}{s^2} - \frac{2}{s} e^{-s} = \frac{2}{s^2} (1 - e^{-s} - se^{-s}) \]

\[ F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \frac{2}{s^2 (1 - e^{-2s})} (1 - e^{-s} - se^{-s}) \]
Example

- Find the initial and final values of the function whose Laplace transform is

\[ H(s) = \frac{20}{(s + 3)(s^2 + 8s + 25)} \]

- Solution:

\[
h(0) = \lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{20s}{(s + 3)(s^2 + 8s + 25)}
\]

\[
= \lim_{s \to \infty} \frac{20/s^2}{(1 + 3/s)(1 + 8/s + 25/s^2)} = \frac{0}{(1+0)(1+0+0)} = 0
\]
Example

\[ h(\infty) = \lim_{s \to 0} sH(s) \]

\[ = \lim_{s \to 0} \frac{20s}{(s + 3)(s^2 + 8s + 25)} \]

\[ = \frac{0}{(0 + 3)(0 + 0 + 25)} = 0 \]
The Inverse Laplace Transform

Steps to Find the Inverse Laplace Transform:

1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Table
Simple Poles

\[ F(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n} \]

\[ k_i = (s + p_i)F(S)\bigg|_{s=p_i} \]

\[ f(t) = (k_1 e^{-p_1t} + k_2 e^{-p_2t} + \cdots + k_n e^{-p_nt})u(t) \]
Repeated Poles

\[ F(s) = \frac{k_1}{(s + p_1)} + \frac{k_2}{(s + p_1)^2} + \cdots + \frac{k_n}{(s + p_1)^n} + F_1(s) \]

\[ f(t) = \left( k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!} t^2 e^{-pt} + \cdots + \frac{k_n}{(n-1)!} t^{n-1} e^{-pt} \right) u(t) + f_1(t) \]
Complex Poles

\[ F(s) = \frac{A_1s + A_2}{s^2 + as + b} + F_1(s) \]

\[ = \frac{A_1s + A_2}{(s + \alpha)^2 + \beta^2} + F_1(s) \]

\[ = \frac{A_1(s + \alpha) + B_1\beta}{(s + \alpha)^2 + \beta^2} + F_1(s) \]

\[ = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1\beta}{(s + \alpha)^2 + \beta^2} + F_1(s) \]

\[ f(t) = (A_1e^{-\alpha t} \cos \beta t + B_1e^{-\alpha t} \cos \beta t)u(t) + f_1(t) \]
Example

- Find the inverse Laplace transform of

\[ F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2 + 4} \]

- Solution:

\[ f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left(\frac{3}{s}\right) - \mathcal{L}^{-1}\left(\frac{5}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2 + 4}\right) \]

\[ = (3 - 5e^{-t} + 3\sin 2t)u(t), \quad t \geq 0 \]
Example

- Find $f(t)$ given that

$$F(s) = \frac{s^2 + 12}{s(s + 2)(s + 3)}$$

- Solution:

$$\frac{s^2 + 12}{s(s + 2)(s + 3)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{s + 3}$$

$$A = sF(s) \bigg|_{s=0} = \frac{s^2 + 12}{(s + 2)(s + 3)} \bigg|_{s=0} = \frac{12}{(2)(3)} = 2$$
Example

\[ B = (s + 2)F(s) \bigg|_{s=-2} = \left. \frac{s^2 + 12}{s(s + 3)} \right|_{s=-2} = \frac{4 + 12}{(-2)(1)} = -8 \]

\[ C = (s + 3)F(s) \bigg|_{s=-3} = \left. \frac{s^2 + 12}{s(s + 2)} \right|_{s=-3} = \frac{9 + 12}{(-3)(-1)} = 7 \]
Example

- Find $v(t)$ given that

$$V(s) = \frac{10s^2 + 4}{s(s + 1)(s + 2)^2}$$

- Solution:

$$V(s) = \frac{10s^2 + 4}{s(s + 1)(s + 2)^2} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{(s + 2)^2} + \frac{D}{s + 2}$$
Example

\[
A = sV(s)\bigg|_{s=0} = \frac{10s^2 + 4}{(s + 1)(s + 2)^2} \bigg|_{s=0} = \frac{4}{(1)(2)^2} = 1
\]

\[
B = (s + 1)V(s)\bigg|_{s=-1} = \frac{10s^2 + 4}{s(s + 2)^2} \bigg|_{s=-1} = \frac{14}{(-1)(1)^2} = -14
\]

\[
C = (s + 2)^2 V(s)\bigg|_{s=-2} = \frac{10s^2 + 4}{s(s + 1)} \bigg|_{s=-2} = \frac{44}{(-2)(-1)} = 22
\]

\[
D = \frac{d}{ds}[(s + 2)^2 V(s)]\bigg|_{s=-2} = \frac{d}{ds}\left(\frac{10s^2 + 4}{s^2 + s}\right)\bigg|_{s=-2} = \frac{(s^2 + s)(20s) - (10s^2 + 4)(2s + 1)}{(s^2 + s)^2}
\]

\[
= \frac{52}{4} = 13
\]
Example

- Find the inverse transform of the frequency-domain function in Example 15.7:

\[ H(s) = \frac{20}{(s + 3)(s^2 + 8s + 25)} \]

- Solution:

\[ s^2 + 8s + 25 = 0 \implies s = -4 \pm j3 \]

We let

\[ H(s) = \frac{20}{(s + 3)(s^2 + 8s + 25)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + 8s + 25} \]
Example

\[ A = (s + 3)H(s)\bigg|_{s=-3} = \frac{20}{s^2 + 8s + 25}\bigg|_{s=-3} = \frac{20}{10} = 2 \]

\[ \frac{20}{75} = \frac{A}{3} + \frac{C}{25} \Rightarrow 20 = 25A + 3C \]

Since \( A = 2 \Rightarrow C = -10. \)

\[ s = 1 \Rightarrow \frac{20}{(4)(34)} = \frac{A}{4} + \frac{B + C}{34} \Rightarrow 20 = 34A + 4B + 4C \]

But \( A = 2, C = -10, \) so that \( B = -2. \)
The **convolution** of two signals consists of time-reversing one of the signals, shifting it, and multiplying it point by point with the second signal, and integrating the product.

\[
y(t) = h(t) * x(t) = \int_0^t x(\lambda)h(t - \lambda)d\lambda
\]

\[
F_1(s)F_2(s) = \mathcal{L}[f_1(t) * f_2(t)]
\]
Example

- Use the Laplace transform to solve the differential equation
  \[
  \frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)
  \]
  subject to
  \[
  v(0) = 1, \quad v'(0) = -2.
  \]

- Solution:
  \[
  [s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}
  \]
Example

\[ v(0) = 1, \quad v'(0) = -2. \]

\[ \Rightarrow s^2 V(s) - s + 2 + 6s V(s) - 6 + 8V(s) = \frac{2}{s} \]

\[ \Rightarrow (s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s} \]

Hence,

\[ V(s) = \frac{s^2 + 4s + 2}{s(s + 2)(s + 4)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{s + 4} \]
Example

where

\[ A = sV(s)\bigg|_{s=0} = \frac{s^2 + 4s + 2}{(s + 2)(s + 4)}\bigg|_{s=0} = \frac{2}{(2)(4)} = \frac{1}{4} \]

\[ B = (s + 2)V(s)\bigg|_{s=-2} = \frac{s^2 + 4s + 2}{s(s + 4)}\bigg|_{s=-2} = \frac{-2}{(-2)(2)} = \frac{1}{2} \]

\[ C = (s + 4)V(s)\bigg|_{s=-4} = \frac{s^2 + 4s + 2}{s(s + 2)}\bigg|_{s=-4} = \frac{2}{(-4)(-2)} = \frac{1}{4} \]
Example

Hence,

\[ V(s) = \frac{1}{4} + \frac{1}{2} \frac{1}{s + 2} + \frac{1}{4} \frac{1}{s + 4} \]

\[ \Rightarrow v(t) = \frac{1}{4} (1 + 2e^{-2t} + e^{-4t})u(t) \]
Example

- Solve for the response $y(t)$ in the following integrodifferential equation.

\[
\frac{dy}{dt} + 5y(t) + 6 \int_0^t y(\tau) d\tau = u(t), \quad y(0) = 2
\]

- Solution:

\[
[sY(s) - y(0)] + 5Y(s) + \frac{6}{s}Y(s) = \frac{1}{s}
\]
Example

\[ y(0) = 2 \]

\[ \Rightarrow Y(s)(s^2 + 5s + 6) = 1 + 2s \]

\[ \Rightarrow Y(s) = \frac{2s + 1}{(s + 2)(s + 3)} = \frac{A}{s + 2} + \frac{B}{s + 3} \]

where

\[ A = (s + 2)Y(s) \bigg|_{s=-2} = \frac{2s + 1}{s + 3} \bigg|_{s=-2} = \frac{-3}{1} = -3 \]

\[ B = (s + 3)Y(s) \bigg|_{s=-3} = \frac{2s + 1}{s + 2} \bigg|_{s=-3} = \frac{-5}{-1} = 5 \]
Example

Thus

\[ Y(s) = \frac{-3}{s + 2} + \frac{5}{s + 3} \]

\[ \Rightarrow y(t) = (-3e^{-2t} + 5e^{-3t})u(t) \]
System Stability

ECE 2221
What is a **Stable** System?

- Ball sitting in valley with *rough* sides. An impulse input will set ball in motion, but the ball *returns to the resting state*

- Ball sitting in valley on *frictionless* surface. An impulse input will cause the ball to be set *in motion forever*

- Ball sitting on *top* of a hill. One small push and the ball keeps *on falling forever*
What is a **Stable** System?

**BIBO Stable** (a.k.a. stable in BIBO sense)
- If bounded inputs (i.e. finite amplitude, no matter how large) produce bounded outputs (i.e. finite output, never go to infinity)

---

**Stable**
- If the input stops and the output slowly decays away, or else the input continues and the output signal behaves itself (doesn’t blow up)

**Marginally Stable**
- If the output remains at a constant size forever even though the input signal has long since stopped

**Unstable**
- If the input stops or remains finite and the output signal continues to grow larger and larger until reaching the infinite proportions.
Conditions for Stability

(1) Impulse Response

\[ \text{BIBO Stability} \iff \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \quad \text{or} \quad \sum_{k=-\infty}^{\infty} |h(k)| < \infty \]

A system is BIBO stable if and only if its impulse response is absolutely summable or integrable.

(2) ROC of System Function

A system is BIBO stable if and only if the ROC includes the \( \omega \)-axis (continuous-time system) or the unit circle (discrete-time system).

For causal system, stability means that all poles are in the shaded area. (locations of zeros do not affect system stability)
(3) Characteristic Equations

- All poles are in the left-half plane.
- This implies that the roots of the characteristic equation (denominator of the system function \( H(s) \)) are all negative, or at least have negative real parts.

<table>
<thead>
<tr>
<th>System Order</th>
<th>Characteristic Equation</th>
<th>Stability Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( s + a = 0 )</td>
<td>( a &gt; 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( s^2 + as + b = 0 )</td>
<td>( a &gt; 0, b &gt; 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( s^3 + as^2 + bs + c = 0 )</td>
<td>( a, b, c &gt; 0 ) and ( ab &gt; c )</td>
</tr>
</tbody>
</table>
Causal, stable systems

The system function of an LTI system, \( H(s) \), can be used to categorize systems.

- \textit{Causal system} \( \Rightarrow \) ROC is to the right of the rightmost pole of \( H(s) \).
- \textit{Causal, stable system} \( \Rightarrow \) ROC is to the right of the rightmost pole of \( H(s) \) and all the poles are in the left-half of the \( s \) plane.

More on the definition of stable systems at a later time.
The transfer function is given inside the block.
The input in this case is $E(s)$.
The output in this case is $C(s)$.
$C(s) = G(s) E(s)$.
Cascaded System

\[ R(s) \xrightarrow{G_1(s)} X_2(s) = G_1(s)R(s) \xrightarrow{G_2(s)} X_1(s) = G_2(s)G_1(s)R(s) \xrightarrow{G_3(s)} C(s) = G_2(s)G_1(s)R(s) \]

\[ R(s) \xrightarrow{G_3(s)G_2(s)G_1(s)} C(s) \]
Parallel System

\[ X_1(s) = R(s)G_1(s) \]
\[ X_2(s) = R(s)G_2(s) \]
\[ X_3(s) = R(s)G_3(s) \]

\[ C(s) = [\pm G_1(s) \pm G_2(s) \pm G_3(s)]R(s) \]
Feedback System

(a) Feedback System

(b) Feedback System

(c) Feedback System
Block Diagram Algebra
Block Diagram Algebra

(a) and (b) diagrams showing signal flow and algebraic manipulations involving $G(s)$ and $R(s)$. The diagrams illustrate the relationships and transformations in the system.
Example 1

- Collapse summing junctions
- Form equivalent cascaded system in the forward path and equivalent parallel system in the feedback path
- Form equivalent feedback system and multiply by cascaded $G_1(s)$
Example 2
Applications of the Laplace Transform
Applications of LT

- Introduction
- Circuit Element Models
- Circuit Analysis
- Transfer Functions
- State Variables
A system is a mathematical model of a physical process relating the input to the output.
Circuit Element Models

- **Steps in Applying the Laplace Transform:**
  1. Transform the circuit from the time domain to the s-domain.
  2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique.
  3. Take the inverse transform of the solution and thus obtain the solution in the time domain.
\[ V(s) = RI(s) \]

\[ I(s) = \frac{1}{sL} V(s) + \frac{i(0^-)}{s} \]

\[ V(s) = \frac{1}{sC} I(s) + \frac{i(0^-)}{s} \]
Resistor:

\[ v(t) = i(t)R \iff V(s) = RI(s) \]
Inductors:

\[ v(t) = L \frac{di}{dt} \iff V(s) = L(sI(s) - i(0^-)) \]

\[ V(s) = sLI(s) - Li(0^-) \]

\[ I(s) = \frac{V(s)}{SL} + \frac{i(0^-)}{s} \]
Inductor

(a) $i(t)$

$+ i(0)$

$v(t)$

$L$

(b) $I(s)$

$+ sL$

$L i(0^-)$

(c) $V(s)$

$sL$

$I(s)$

$\frac{i(0^-)}{s}$
Capacitors:

\[ i(t) = C \frac{dv}{dt} \iff I(s) = C(sV(s) - v(0^-)) \]

\[ I(s) = sCV(s) - Cv(0^-) \]

\[ V(s) = \frac{I(s)}{SC} + \frac{v(0^-)}{s} \]
Capacitor

(a) $i(t)$

$+ \quad + \quad + \quad C \quad +$

$v(t) \quad v(0) \quad - \quad -$

(b) $I(s)$

$+ \quad + \quad + \quad \frac{1}{sC} \quad -$

$V(s) \quad \frac{v(0)}{s} \quad -$

(c) $I(s)$

$+ \quad + \quad \frac{1}{sC} \quad -$

$V(s) \quad Cv(0)$
Circuit with Zero Initials

\[ Z(s) = \frac{V(s)}{I(s)} \]

Resistor: \[ Z(s) = R \]
Inductor: \[ Z(s) = sL \]
Capacitor: \[ Z(s) = \frac{1}{sC} \]
\[ Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \]

\[ \mathcal{L}[av(t)] = aV(s) \]
\[ \mathcal{L}[ai(t)] = aI(s) \]

### Table 16.1

**Impedance of an element in the s-domain.**

<table>
<thead>
<tr>
<th>Element</th>
<th>( \mathcal{Z}(s) = V(s)/I(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>( R )</td>
</tr>
<tr>
<td>Inductor</td>
<td>( sL )</td>
</tr>
<tr>
<td>Capacitor</td>
<td>( 1/sC )</td>
</tr>
</tbody>
</table>

*Assuming zero initial conditions*
Example

- Find $v_o(t)$, assuming zero initial conditions.
Example

\[ \frac{1}{s} \quad + \quad \frac{3}{s} \quad = \quad 1 \Omega \quad + \quad 5 \Omega \quad s \quad V_o(s) \quad + \quad \frac{1}{s} \quad I_1(s) \quad \Rightarrow \quad u(t) \quad \Rightarrow \quad \frac{1}{s} \quad \frac{1}{1 \text{ H}} \quad \Rightarrow \quad sL = s \quad \frac{1}{3 \text{ F}} \quad \Rightarrow \quad \frac{1}{sC} = \frac{3}{s} \]
Example

For mesh 1, \( \frac{1}{s} = \left(1 + \frac{3}{s}\right)I_1 - \frac{3}{s}I_2 \)

For mesh 2, \( 0 = -\frac{3}{s}I_1 + \left(s + 5 + \frac{3}{s}\right)I_2 \) \( \Rightarrow \) \( I_1 = \frac{1}{3}(s^2 + 5s + 3)I_2 \)

\( \Rightarrow \frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2 \)

\( \Rightarrow 3 = (s^3 + 8s^2 + 18s)I_2 \)

\( \Rightarrow I_2 = \frac{3}{s^3 + 8s^2 + 18s} \)
Example

\[ V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s + 4)^2 + (\sqrt{2})^2} \]

\[ \Rightarrow v_o(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin(\sqrt{2}t)V, \quad t \geq 0 \]
• Find $v_o(t)$. Assume $v_o(0)=5$ V.
Example

\[
\frac{10}{s + 1} - \frac{V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}
\]

\[
\Rightarrow \frac{1}{s + 1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10} V_o (s + 2)
\]
\[
\frac{10}{s+1} + 25 = V_o(s + 2)
\]

\[
\Rightarrow V_o = \frac{25s + 35}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}
\]

where

\[
A = (s + 1)V_o(s)\bigg|_{s=-1} = \frac{25s + 35}{s + 2}\bigg|_{s=-1} = \frac{10}{1} = 10
\]

\[
B = (s + 2)V_o(s)\bigg|_{s=-2} = \frac{25s + 35}{s + 1}\bigg|_{s=-2} = -\frac{15}{-1} = 15
\]
Example

Thus

\[ V_o(s) = \frac{10}{s + 1} + \frac{15}{s + 2} \]

\[ \Rightarrow v_o(t) = (10e^{-t} + 15e^{-2t})u(t) \text{ V} \]
Example

- In the circuit, the switch moves from position $a$ to position $b$ at $t = 0$. Find $i(t)$ for $t > 0$. 

![Diagram of the circuit with switch moved from position $a$ to position $b$.]({})
\[ I(s)(R + sL) - LI_o - \frac{V_o}{s} = 0 \]

\[ \Rightarrow I(s) = \frac{LI_o}{R + sL} + \frac{V_o}{s(R + sL)} = \frac{I_o}{s + R/L} - \frac{V_o / L}{s(s + R/L)} \]

\[ I(s) = \frac{I_o}{s + R/L} + \frac{V_o / R}{s} - \frac{V_o / R}{(s + R/L)} \]

\[ \Rightarrow i(t) = \left( I_o - \frac{V_o}{R} \right) e^{-t/\tau} + \frac{V_o}{R}, \quad t \geq 0 \quad \text{where } \tau = \frac{R}{L} \]
Example

The initial value \( i(\infty) = \frac{V_o}{R} \),

\[
\lim_{s \to 0} sI(s) = \lim_{s \to 0} \left( \frac{sI_o}{s + R/L} + \frac{V_o/L}{s + R/L} \right) = \frac{V_o}{R}
\]

\[\Rightarrow i(t) = I_o e^{-t/\tau} + \frac{V_o}{R} (1 - e^{-t/\tau}), \quad t \geq 0\]

In the initial condition, \( I_o = 0 \),

\[\Rightarrow i(t) = \frac{V_o}{R} (1 - e^{-t/\tau}), \quad t \geq 0\]
• Remember, equivalent circuits, with capacitors and inductors, only exist in the s-domain; they cannot be transformed back into the time domain.
Example

- Find the value of the voltage across the capacitor assuming that the value of \( v_s(t) = 10u(t) \) V and assume that at \( t = 0 \), -1 A flows through the inductor and +5 V is across the capacitor.
\[
\begin{align*}
\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} - \frac{i(0)}{s} + \frac{V_1 - [v(0)/s]}{1/(0.1s)} &= 0 \\
\Rightarrow 0.1 \left( s + 3 + \frac{2}{s} \right)V_1 &= \frac{3}{s} + \frac{1}{s} + 0.5 \\
\text{where } v(0) &= 5 \text{ V and } i(0) = -1 \text{ A} \\
\Rightarrow (s^2 + 3s + 2)V_1 &= 40 + 5s \\
\Rightarrow V_1 &= \frac{40 + 5s}{(s + 1)(s + 2)} = \frac{35}{s + 1} - \frac{30}{s + 2} \\
\text{So, } v_1(t) &= (35e^{-t} - 30e^{-2t})u(t) \text{ V}
\end{align*}
\]
Example

- Assume that there is no initial energy stored in the circuit at $t = 0$ and that $i_s = 10 \ u(t)$. (a) Find $v_o(s)$ using Thevenin’s theorem. (b) Apply the initial- and final-value theorems to find $v_o(0^+)$ and $v_o(\infty)$. (c) Determine $v_o(t)$. 

![Circuit Diagram]
Example

(a)

Since \( I_x = 0 \),

\[
V_{oc} = V_{TH} = 5 \left( \frac{10}{s} \right) = \frac{50}{s}
\]

\[
I_{sc} = I_x = V_1 / 2s
\]

\[
\Rightarrow - \frac{10}{s} + \frac{(V_1 - 2I_x)}{5} - 0 + \frac{V_1 - 0}{2s} = 0
\]

\[
\Rightarrow V_1 = \frac{100}{2s + 3}
\]
Hence,

\[ I_{sc} = \frac{V_1}{2s} = \frac{100/(2s + 3)}{2s} = \frac{50}{s(2s + 3)} \]

\[ Z_{TH} = \frac{V_{oc}}{I_{sc}} = \frac{50/s}{50/[s(2s + 3)]} = 2s + 3 \]

\[ \Rightarrow V_o = \frac{5}{5 + Z_{TH}} V_{TH} = \frac{5}{5 + 2s + 3\left(\frac{50}{s}\right)} = \frac{250}{s(2s + 8)} = \frac{125}{s(s + 4)} \]
(b) Using the initial-value theorem we find

\[
\nu_0(0) = \lim_{s \to \infty} s V_0(s) = \lim_{s \to \infty} \frac{125}{s + 4} = \lim_{s \to \infty} \frac{125/s}{1 + 4/s} = \frac{0}{1} = 0
\]

Using the final-value theorem we find

\[
\nu_0(\infty) = \lim_{s \to 0} s V_0(s) = \lim_{s \to 0} \frac{125}{s + 4} = \frac{125}{4} = 31.25 \text{ V}
\]
Example
(c) By partial fraction,

\[ V_o = \frac{125}{s(s + 4)} = \frac{A}{s} + \frac{B}{s + 4} \]

\[ A = sV_o(s) \bigg|_{s=0} = \frac{125}{s+4} \bigg|_{s=0} = 31.25 \]

\[ B = (s + 4)V_o(s) \bigg|_{s=-4} = \frac{125}{s} \bigg|_{s=-7} = -31.25 \]

\[ V_o = \frac{31.25}{s} - \frac{31.25}{s + 4} \]

\Rightarrow v_o(t) = 31.25(1 - e^{-4t})u(t) \text{ V}
The transfer function $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

$$H(s) = \frac{Y(s)}{X(s)}$$
\[ H(s) = \text{Voltage gain} = \frac{V_o(s)}{V_i(s)} \]

\[ H(s) = \text{Current gain} = \frac{I_o(s)}{I_i(s)} \]

\[ H(s) = \text{Impedance} = \frac{V(s)}{I(s)} \]

\[ H(s) = \text{Admittance} = \frac{I(s)}{V(s)} \]
\[ Y(s) = H(s)X(s) \]
\[ x(t) = \delta(t), \text{ so that } X(s) = 1 \]
\[ \Rightarrow Y(s) = H(s) \quad \text{or} \quad y(t) = h(t) \]
\[ \text{where } h(t) = \mathcal{L}^{-1}[H(s)] \]
Example

- The output of a linear system is \( y(t) = 10e^{-t} \cos 4t \ u(t) \) when input is \( x(t) = e^{-t}u(t) \). Find the transfer function of the system and its impulse response.

**Solution:**

- If \( x(t) = e^{-t}u(t) \) and \( y(t) = 10e^{-t} \cos 4t \ u(t) \), then

\[
X(s) = \frac{1}{s + 1} \quad \text{and} \quad Y(s) = \frac{10(s + 1)}{(s + 1)^2 + 4^2}
\]

Hence,

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{10(s + 1)^2}{(s + 1)^2 + 16} = \frac{10(s^2 + 2s + 1)}{s^2 + 2s + 17}
\]
To find $h(t)$,

$$H(s) = 10 - 40 \frac{4}{(s + 1)^2 + 4^2}$$

$$\Rightarrow h(t) = 10\delta(t) - 40e^{-t} \sin 4tu(t)$$
Example

- Determine the transfer function $H(s) = \frac{V_o(s)}{I_o(s)}$ of the circuit
Example

\[ I_2 = \frac{(s + 4)I_o}{s + 4 + 2 + 1/2s} \]

But

\[ V_o = 2I_2 = \frac{2(s + 4)I_o}{s + 5 + 1/2s} \]

Hence,

\[ H(s) = \frac{V_o(s)}{I_o(s)} = \frac{4s(s + 4)}{2s^2 + 12s + 1} \]
For the s-domain circuit in Fig. 16.19, find: (a) the transfer function \( H(s) = \frac{V_o}{V_i} \), (b) the impulse response, (c) the response when \( v_i(t) = u(t) \, \text{C} \), (d) the response when \( v_i(t) = 8\cos2t \, \text{V} \).
Example

(a) \[ V_o = \frac{1}{s + 1} V_{ab}, \quad \text{but} \]

\[ V_{ab} = \frac{1}{1 + 1/(s + 1)} V_i = \frac{(s + 1)/(s + 2)}{1 + (s + 1)/(s + 2)} V_i \Rightarrow V_{ab} = \frac{s + 1}{2s + 3} V_i \]

So, \[ V_o = \frac{V_i}{2s + 3}. \quad \text{Thus,} \quad H(s) = \frac{V_o}{V_i} = \frac{1}{2s + 3} \]

(b) \[ H(s) = \frac{1}{2 s + \frac{3}{2}} \]

\[ \Rightarrow h(s) = \frac{1}{2} e^{-3t/2} u(t) \]
(c) \[ v_i(t) = u(t), \quad V_i(s) = 1/s \]

\[ V_o(s) = H(s)V_i(s) = \frac{2}{2s(s + \frac{3}{2})} = \frac{A}{s} + \frac{B}{s + \frac{3}{2}} \]

where

\[ A = sV_o(s) \bigg|_{s=0} = \frac{1}{2(s + \frac{3}{2})} \bigg|_{s=0} = \frac{1}{3} \]

\[ B = \left( s + \frac{3}{2} \right)V_o(s) \bigg|_{s=-3/2} = \frac{1}{2s} \bigg|_{s=-3/2} = -\frac{1}{3} \]
Example

Hence, for \( v_i(t) = u(t) \),

\[
V_0(s) = \frac{1}{3} \left( \frac{1}{s} - \frac{1}{s + \frac{3}{2}} \right) \Rightarrow v_o(t) = \frac{1}{3} (1 - e^{-3t/2}) u(t) \text{ V}
\]

(d) When \( v_i(t) = 8 \cos 2t \), then \( V_i(s) = \frac{8s}{s^2 + 4} \), and

\[
V_o(s) = H(s)V_i(s) = \frac{4s}{(s + \frac{3}{2})(s^2 + 4)} = \frac{A}{s + \frac{3}{2}} + \frac{Bs + C}{s^2 + 4}
\]

where

\[
A = \left. \left( s + \frac{3}{2} \right) V_o(s) \right|_{s=-3/2} = \left. \frac{4s}{s^2 + 4} \right|_{s=-3/2} = -\frac{24}{25}
\]
Example

\[ 4s = A(s^2 + 4) + B\left(s^2 + \frac{3}{2}s\right) + C\left(s + \frac{3}{2}\right) \]

Equating coefficients,

Constants : \[ 0 = 4A + \frac{3}{2}C \quad \Rightarrow \quad C = -\frac{8}{3}A \]

\[ s : \quad 4 = \frac{3}{2}B + C \]

\[ s^2 : \quad 0 = A + B \quad \Rightarrow \quad B = -A \]
Example

Solving these gives $A = -24/25$, $B = 24/25$, $C = 64/25$. Hence, for $v_i(t) = 8\cos 2t \text{ V}$,

$$V_o(s) = \frac{-24}{25} + \frac{24}{25} \frac{s}{s^2 + \frac{3}{2}} + \frac{32}{25} \frac{2}{s^2 + 4}$$

$$\Rightarrow v_o(t) = \frac{24}{25} \left( -e^{-3t/2} + \cos 2t + \frac{4}{3} \sin 2t \right) u(t) \text{ V}$$
State Variables
• A **state variable** is a physical property that characterizes the state of a system, regardless of how the system got to that state.

\[ \dot{x} = Ax + Bz \]

where

\[ \dot{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \text{state vector representing } n \text{ state vectors.} \]
\[ \dot{x} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} \]

and

\[ z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \text{input vector representing } m \text{ input.} \]
\[
\dot{x} = Ax + Bz \\
y = Cx + Dz
\]

where

\[
y(t) = \begin{bmatrix}
y_1(t) \\
y_2(t) \\
\vdots \\
y_p(t)
\end{bmatrix} = \text{the output vector representing } p \text{ outputs}
\]

\(A\) and \(B\) are respectively \(n \times n\) and \(n \times m\) matrices. \(C\) and \(D\) are respectively \(p \times n\) and \(p \times m\) matrices.
\[ sX(s) = AX(s) + BZ(s) \quad \Rightarrow \quad (sI - A)X(s) = BZ(s) \]

\[ \Rightarrow X(s) = (sI - A)^{-1}BZ(s) \]

where \( I \) is the identity matrix.

\[ Y(s) = CX(s) + DZ(s) \]

\[ \Rightarrow H(s) = \frac{Y(s)}{Z(s)} = C(sI - A)^{-1}B + D \]

where \( A = \) system matrix

\( B = \) input coupling matrix

\( C = \) output matrix

\( D = \) feedforward matrix
Steps to Apply the State Variable Method to Circuit Analysis:

1. Select the inductor current $i$ and capacitor voltage $v$ as the state variables, making sure they are consistent with the passive sign convention.

2. Applying KCL and KVL to the circuit and obtain circuit variables (voltage and currents) in terms of the state variables. This should lead to a set of first-order differential equations necessary and sufficient to determine all state variables.

3. Obtain the output equation and put the final result in state-space representation.
Example 16.10

- Find the state-space representation of the circuit. Determine the transfer function of the circuit when $v_s$ is the input and $i_x$ is the output. Take $R = 1\Omega$, $C = 0.25$ F, and $L = 0.5$ H.
Example

\[ v_L = L \frac{di}{dt} \]

\[ i_C = C \frac{dv}{dt} \]

Applying KCL at node 1 gives

\[ i = i_x + i_C \rightarrow C \frac{dv}{dt} = i - \frac{V}{R} \]

\[ \Rightarrow \dot{v} = -\frac{v}{RC} + \frac{i}{C} \]

\[ v_s = v_L + v \rightarrow L \frac{di}{dt} = -v + v_s \]

\[ i = -\frac{v}{L} + \frac{v_s}{L}, \quad i = \frac{v}{R} \]

\[ \begin{bmatrix} \dot{v} \\ i \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_s \]

\[ i_x = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} \]
If $R = 1$, $C = \frac{1}{4}$, and $L = \frac{1}{2}$, we obtain

$$A = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{1}{R} & 0 \\ \frac{1}{R} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & 4 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s - 4 & -4 \\ 2 & s \end{bmatrix}$$
Example

- Taking the inverse of this gives

\[(sI - A)^{-1} = \frac{\text{adjoint of } A}{\text{determinant of } A} = \begin{bmatrix} s & 4 \\ -2 & s + 4 \end{bmatrix}\]

\[
\Rightarrow H(s) = C(sI - A)^{-1}B
\]

\[
\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 4 \\ -2 & s + 4 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 2s + 8 \end{bmatrix} = \frac{8}{s^2 + 4s + 8}
\]
Consider the circuit in Fig. 16.24, which may be regarded as a two-input, two-output. Determine the state variable model and find the transfer function of the system.
Example

\[-v_s + i_1 + \frac{1}{6}i = 0 \Rightarrow i = 6v_s - 6i\]

\[v_s = i_1 + v_o + v\]

- At node 1, KCL gives

\[i_1 = i + \frac{v_0}{2} \rightarrow v_0 = 2(i_1 - i)\]

\[v_s = 3i + v - 2i \rightarrow i_1 = \frac{2i - v + v_s}{3} \Rightarrow i = 2v - 4i + 4v_s\]

- At node 2,

\[\frac{v_o}{2} = \frac{1}{3} \dot{v} + i_o \rightarrow \dot{v} = \frac{3}{2}v_o - 3i_o \Rightarrow i_o = \frac{v - v_i}{3}\]

\[v_o = 2\left(\frac{2i - v + v_s}{3} - i\right) = -\frac{3}{2}(v + i - v_s)\]
\[ \dot{v} = -2v - i + v_s + v_1 \]

\[ \Rightarrow \]

\[ \begin{bmatrix} \dot{v} \\ i \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ v_i \end{bmatrix} \]

\[ \begin{bmatrix} v_o \\ i_o \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} v_s \\ v_i \end{bmatrix} \]
Example

Assume we have a system where the output is $y(t)$ and the input is $z(t)$. Let the following differential equation describe the relationship between the input and output:

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2 y(t) = 5z(t)$$

Obtain the state model and the transfer function of the system.
Let \( x_1 = y(t), \dot{x} = \dot{y}(t) \)

Now let \( x_2 = \dot{x}_1 = \dot{y}(t) \)

\[
\dot{x}_2 = \ddot{y}(t) = -2y(t) - 3\dot{y}(t) + 5z(t) = -2x_1 - 3x_2 + 5z(t)
\]

\[
\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} z(t)
\]

\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

\[
\Rightarrow sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s + 3 \end{bmatrix}
\]
Example

The inverse is \((sI - A)^{-1} = \frac{\begin{bmatrix} s + 3 & 1 \\ -2 & s \end{bmatrix}}{s(s + 3) + 2}\)

The transfer function is

\[H(s) = C(sI - A)^{-1}B = \frac{(1 \ 0)\begin{bmatrix} s + 3 & 1 \\ -2 & s \end{bmatrix}(0 \ 5)}{s(s + 3) + 2}\]

\[= \frac{(1 \ 0)\begin{bmatrix} 5 \\ 5s \end{bmatrix}}{s(s + 3) + 2} = \frac{5}{(s + 1)(s + 2)}\]
Example

\[ (s^2 + 3s + 2)Y(s) = 5Z(s) \]

\[ \rightarrow H(s) = \frac{Y(s)}{Z(s)} = \frac{5}{s^2 + 3s + 2} \]